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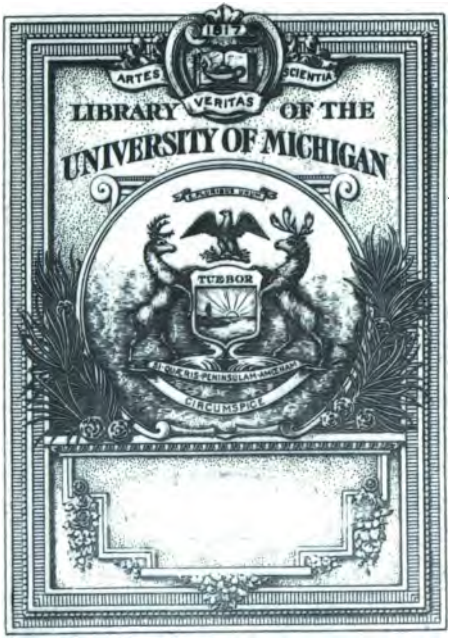

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PRACTICAL TEXT-BOOK

ON

PLANE AND SPHERICAL TRIGONOMETRY.

BY

WEBSTER WELLS, S. B.,
ASSISTANT PROFESSOR OF MATHEMATICS IN THE MASSACHUSETTS INSTITUTE OF TECHNOLOGY.

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PREFACE.

THIS work is intended as a practical text-book on Plane and Spherical Trigonometry, adapted to the needs of High Schools and Colleges. While considerable space has been devoted to the development of the theoretical portions of the subject, the practical applications of the formulæ have not been neglected, and in the chapters on the Solution of Triangles will be found a carefully selected assortment of exercises and examples, arranged with the view of giving the student a thorough drill in numerical computation.

Chapter II. is devoted to the functions of acute angles, the definitions being given by aid of the right triangle; a number of numerical and other exercises serve to give familiarity with them, and the usual fundamental relations are deduced. In Chapter III. the principles of the rectangular co-ordinates of a point in a plane are employed to give general definitions of the functions, applicable to angles of any magnitude, positive or negative, and the fundamental relations are proved to hold universally.

Before taking up the solution of triangles, a full exposition of the theory and applications of Logarithms is given, including the calculation and use of the tables of the logarithms of numbers and of the Trigonometrical functions.

The chapters on the Solution of Cubic Equations, and on De Moivre's Theorem, will serve to introduce the student to some of the higher applications of Trigonometry, which are not usually found in elementary text-books.

Part II. is devoted to Spherical Trigonometry; the usual formulæ for Right and Oblique Spherical Triangles are proved, and methods are given for the solutions of all cases. At the end of the book will be found a few examples showing the application of the results to some of the simpler problems of Geodesy and Navigation.

The author has made use of a number of treatises in the preparation of this volume, and would mention his special indebtedness to those of Todhunter, Snowball, and Chauvenet. He would also acknowledge his obligations for a number of suggestions from teachers of experience, which add materially to the value of the work.

WEBSTER WELLS.

BOSTON, 1883.

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PLANE AND SPHERICAL TRIGONOMETRY.

PART I.

PLANE TRIGONOMETRY.

I. DEFINITIONS; MEASUREMENT OF ANGLES.

1. **Trigonometry** is that branch of mathematics in which algebraical processes are employed to determine the relations between angles, and to subject angles and triangles to computation.

In *Plane Trigonometry* we are concerned with plane angles and triangles.

2. An angle is measured by expressing the number of times it contains a certain angle adopted arbitrarily as the unit of measurement.

The usual unit of angular measurement is the *degree*, or the ninetieth part of a right angle. To express fractional parts of the unit, the degree is divided into sixty equal parts called *minutes*, and the minute into sixty equal parts called *seconds*. This method of measurement is called the *sexagesimal*.

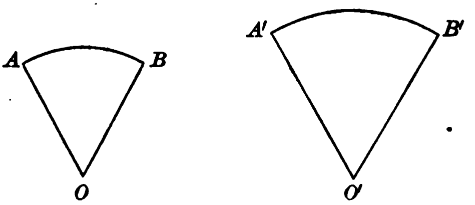
3. Degrees, minutes, and seconds are denoted by the symbols $^\circ$, $'$, $''$, respectively; thus, $43^\circ\,22'\,37''$ denotes an angle of 43 degrees, 22 minutes, and 37 seconds.

4. Another method of measurement, known as the *centesimal*, was proposed in France. In this the unit is the hundredth part of a right angle, and is called a *grade;* the grade is divided into one hundred equal parts called *minutes*, and the minute into one hundred equal parts called *seconds*.

This notation has obvious advantages; but, owing to the fact that the sexagesimal method was the first employed, and that all the mathematical tables had been calculated with reference to it, the centesimal method has not come into practical use.

Grades and centesimal minutes and seconds are denoted by the symbols ^{g}, $^\backprime$, $^{\backprime\backprime}$, respectively; thus, $9^g\ 13^\backprime\ 46^{\backprime\backprime}$ signifies 9 grades, 13 minutes, 46 seconds.

5. A third method of measuring angles, and one of great importance, is that known as the *circular* method.



Let OAB and $O'A'B'$ be two similar sectors; *i.e.*, the angles AOB and $A'O'B'$ being equal. Then, by Geometry,

$$\text{Arc } AB : \text{Arc } A'B' = OA : O'A',$$

whence,
$$\frac{\text{Arc } AB}{OA} = \frac{\text{Arc } A'B'}{O'A'}.$$

Therefore, the fraction $\dfrac{\text{arc}}{\text{radius}}$ is a function of the angle which is independent of the length of the radius, and may consequently be used as a measure of the magnitude of the angle. This is called the circular measure of the angle; that is,

The circular measure of an angle is the ratio of the arc which subtends it to the radius.

To illustrate, the circular measure of a right angle is the ratio of one-fourth of the circumference to the radius; but the circumference of a circle is equal to the radius multiplied by 2π, where $\pi = 3.14159265\cdots$ Hence, if R denote the radius, we have

$$\text{Circular measure of } 90^\circ = \frac{\frac{1}{4} \text{ of } 2\pi R}{R} = \frac{\pi}{2}.$$

6. We found in the preceding article that the circular measure of an angle of 90° is $\frac{\pi}{2}$. Hence the circular measure of an angle of 180° is π; of 60°, $\frac{\pi}{3}$; of 45°, $\frac{\pi}{4}$; of 30°, $\frac{\pi}{6}$; etc.

Thus, an angle of any number of degrees may be expressed in circular measure by finding its ratio to 180°, and multiplying the result by π.

For example, the circular measure of 115° is $\frac{23\pi}{36}$, since 115° is $\frac{23}{36}$ of 180°.

Conversely, an angle expressed in circular measure may be reduced to degrees by multiplying by 180° and dividing by π, or, more briefly, by substituting 180° for π.

For example, the angle $\frac{7\pi}{15} = \frac{7}{15}$ of $180^\circ = 84^\circ$.

7. In the circular method we may have such expressions as "the angle $\frac{2}{3}$," "the angle 1," etc. These refer to the unit of circular measure; that is, the angle $\frac{2}{3}$ means an angle whose arc is $\frac{2}{3}$ of the radius. The angle 1, that is, the angle whose arc is equal to the radius, or the *unit* of circular measure, reduced to degrees by the rule of Art. 6, gives

$$\frac{180^\circ}{\pi} = \frac{180^\circ}{3.14159265\cdots} = 57.2957795^\circ\cdots$$

We may then modify our rule as follows:

The sexagesimal measure of an angle is obtained from its circular measure by multiplying by $57.2957795^\circ\cdots$

For example,

$$\text{The angle } \frac{2}{3} = \frac{2}{3} \times 57.2957795^\circ\cdots = 38.1971863^\circ\cdots$$
$$= 38^\circ\, 11'\, 49.87068''\cdots$$

EXAMPLES.

Express the following angles in circular measure:

1. 135°. 3. $11^\circ\,15'$. 5. $29^\circ\,15'$. 7. $128^\circ\,34\frac{2}{7}'$.

2. 270°. 4. $37^\circ\,30'$. 6. $174^\circ\,22'\,30''$. 8. $92^\circ\,48'\,45''$.

Express the following angles in sexagesimal measure:

$28^\circ\,38'\,52.4''$ 9. $\frac{1}{2}$. 11. $\frac{37\pi}{30}$. 13. $\frac{3}{4}$. 15. $\frac{2\pi-1}{3}$.

$42^\circ\,58'\,18.6''$ $100^\circ\,54'\,5.1''$

10. $\frac{3\pi}{5}$. 12. $\frac{5\pi}{4}$. 14. 2. 16. $\frac{\pi-1}{4}$.

$114^\circ\,35'\,29.6''$ $30^\circ\,40'\,33.8''$

8. The **Complement** of an angle is the remainder obtained by subtracting it from a right angle.

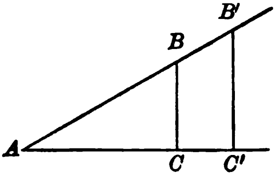
The **Supplement** of an angle is the remainder obtained by subtracting it from two right angles.

To illustrate, the complement of 70° is 20°, and the supplement is 110°.

II. THE TRIGONOMETRICAL FUNCTIONS.

FUNCTIONS OF ACUTE ANGLES.

9. Let BAC be any acute angle.



From any point on either side, as B, draw a line BC perpendicular to the other side, thus forming a right triangle

ABC. The fraction $\frac{BC}{AB}$ is then called the *sine* of the angle BAC; that is, in a right triangle,

The sine of either acute angle is equal to the ratio of the side opposite the angle to the hypothenuse.

It is important to observe that the value of the sine of an angle depends solely on the magnitude of the angle, and not at all on the length of the sides of the right triangle which contains it.

For example, if we had taken our fixed point at B' and had drawn the perpendicular $B'C'$, we should have had the sine of the angle BAC equal to $\frac{B'C'}{A'B'}$. But the right triangles ABC and $AB'C'$, having the angle A in common, are similar; and, by Geometry, the ratio of BC to AB is equal to the ratio of $B'C'$ to AB'. Thus, the two values obtained for the sine of BAC are seen to be equal.

10. Besides the sine, there are five other functions of the angle, whose definitions are similar in form.

In any right triangle,

The cosine of either acute angle is equal to the ratio of the side adjacent to the hypothenuse.

The tangent is equal to the ratio of the side opposite to the side adjacent.

The cotangent is equal to the ratio of the side adjacent to the side opposite.

The secant is equal to the ratio of the hypothenuse to the side adjacent.

The cosecant is equal to the ratio of the hypothenuse to the side opposite.

We have also the following definitions:

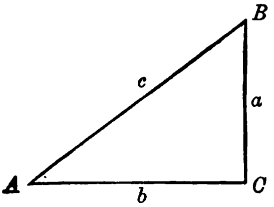
The versed sine of an angle is equal to unity minus the cosine of the angle.

The coversed sine is equal to unity minus the sine.

The suversed sine is equal to unity plus the cosine.

These nine are called the *trigonometrical functions* or *trigonometrical ratios* of the angle. In each case it is evident, as in the previous article, that the value of the function depends only on the magnitude of the angle, and is independent of the dimensions of the right triangle which contains it.

11. It is customary, in dealing with triangles in Trigonometry, to denote the angles by capital letters, and the sides opposite to them by the corresponding small letters.



Applying the definitions to the angles A and B, and adopting the usual abbreviations for the names of the functions, we obtain:

$$\left.\begin{array}{ll}
\sin A = \frac{a}{c} & \sin B = \frac{b}{c} \\
\cos A = \frac{b}{c} & \cos B = \frac{a}{c} \\
\tan A = \frac{a}{b} & \tan B = \frac{b}{a} \\
\cot A = \frac{b}{a} & \cot B = \frac{a}{b} \\
\sec A = \frac{c}{b} & \sec B = \frac{c}{a} \\
\operatorname{cosec} A = \frac{c}{a} & \operatorname{cosec} B = \frac{c}{b} \\
\operatorname{vers} A = 1 - \frac{b}{c} & \operatorname{vers} B = 1 - \frac{a}{c} \\
\operatorname{covers} A = 1 - \frac{a}{c} & \operatorname{covers} B = 1 - \frac{b}{c} \\
\operatorname{suvers} A = 1 + \frac{b}{c} & \operatorname{suvers} B = 1 + \frac{a}{c}
\end{array}\right\} \qquad (1)$$

12. From the first two formulæ of each column we obtain

$$a = c \sin A \qquad b = c \sin B$$
$$b = c \cos A \qquad a = c \cos B$$

The principle here expressed may be enunciated as follows:

In any right triangle either side about the right angle is equal to the hypothenuse multiplied by the sine of the opposite angle or by the cosine of the adjacent angle.

We also have

$$a = b \tan A \qquad b = a \tan B$$
$$b = a \cot A \qquad a = b \cot B$$

That is,

In any right triangle either side about the right angle is equal to the tangent of the opposite angle, or the cotangent of the adjacent angle, multiplied by the other side.

13. By inspection of the formulæ of Art. 11, we deduce the following results:

$$\left.\begin{array}{ll} \sin A = \cos B & \sin B = \cos A \\ \tan A = \cot B & \tan B = \cot A \\ \sec A = \operatorname{cosec} B & \sec B = \operatorname{cosec} A \\ \operatorname{vers} A = \operatorname{covers} B & \operatorname{vers} B = \operatorname{covers} A \end{array}\right\} \quad (2)$$

The important principle here illustrated may be thus stated:

The sine, tangent, secant, and versed sine of an angle are respectively the cosine, cotangent, cosecant, and coversed sine of the complement of the angle.

It is from this circumstance that the names *co*-sine, *co*-tangent, etc., were derived; and the principle enunciated has been adopted by some writers as the definition of the complementary functions.

14. We also obtain from Art. 11 the results

$$\left.\begin{aligned}\sin A \operatorname{cosec} A &= \frac{a}{c}\times\frac{c}{a}=1\\ \cos A \sec A &= \frac{b}{c}\times\frac{c}{b}=1\\ \tan A \cot A &= \frac{a}{b}\times\frac{b}{a}=1\end{aligned}\right\} \quad (3)$$

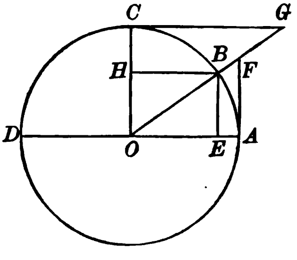
These may be written as follows:

$$\left.\begin{aligned}\sin A &= \frac{1}{\operatorname{cosec} A} & \tan A &= \frac{1}{\cot A} & \sec A &= \frac{1}{\cos A}\\ \cos A &= \frac{1}{\sec A} & \cot A &= \frac{1}{\tan A} & \operatorname{cosec} A &= \frac{1}{\sin A}\end{aligned}\right\} (4)$$

The principle involved is stated thus:

The sine of an angle is the reciprocal of the cosecant, the tangent is the reciprocal of the cotangent, and the secant is the reciprocal of the cosine.

15. The earlier writers on Trigonometry considered the sine, tangent, etc., as functions of an *arc* instead of an *angle*, and they were defined as *lines* instead of *ratios*.



To illustrate, let AB be any arc of a circle whose centre is at O. Draw the radii OA, OB, and OC perpendicular to OA. Draw BE and AF perpendicular to OA, and CG and BH perpendicular to OC, and produce OB to meet CG at G. Then, according to the old definitions, BE was called the sine of the arc AB, OE the cosine, AF the tangent, CG the cotangent, OF the secant, OG the cosecant, AE the versed sine, CH the coversed sine, and DE the suversed sine.

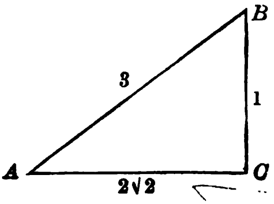
It will be seen that, if the radius of the circle be taken as unity, these definitions give the same numerical values for the functions of the arc AB as our former definitions give for the angle at the centre, AOB, which is subtended by the arc.

Thus, by the definitions, the sine of the arc AB is BE, and in the right triangle OBE the sine of the angle AOB is $\frac{BE}{OB}$; and the two results are evidently equal if OB is unity. Again, the tangent of the arc is AF, and in the right triangle AOF the tangent of the angle AOB is $\frac{AF}{OA}$; results which are equal if OA is unity. In a similar manner it may be shown that the same is true for the other functions.

The method of ratios has been adopted, in preference to the other, by the majority of modern writers; but the student will find no difficulty in passing, if necessary, from the old system to the new; the rule being, as is evident from the illustrations above, to divide the functions of the arc by the radius in order to obtain the corresponding functions of the angle.

16. The Pythagorean proposition affords a simple method for finding the values of the other eight functions of an acute angle, when the value of any one is given.

1. Given $\operatorname{cosec} A = 3$; find the values of the remaining functions.



We may write the equation $\operatorname{cosec} A = \frac{3}{1}$.

Since the cosecant is the ratio of the hypothenuse to the side opposite, we may regard our value as having been taken from a right triangle whose hypothenuse is 3 units in length, and the side opposite the angle A, 1 unit.

Now, by Geometry, $AC^2 + BC^2 = AB^2$.

Hence, $AC = \sqrt{AB^2 - BC^2} = \sqrt{9-1} = \sqrt{8} = 2\sqrt{2}$.

Then, by our definitions, we obtain

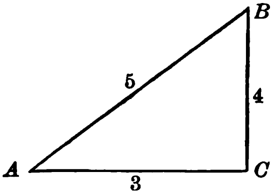
$$\sin A = \frac{1}{3} \qquad \cos A = \frac{2\sqrt{2}}{3}$$

$$\tan A = \frac{1}{2\sqrt{2}} \qquad \cot A = 2\sqrt{2}$$

$$\sec A = \frac{3}{2\sqrt{2}} \qquad \operatorname{covers} A = 1 - \frac{1}{3} = \frac{2}{3}$$

$$\operatorname{vers} A = 1 - \frac{2\sqrt{2}}{3} \qquad \operatorname{suvers} A = 1 + \frac{2\sqrt{2}}{3}$$

2. Given $\operatorname{vers} A = \frac{2}{5}$; find the values of the remaining functions.



Since $\operatorname{vers} A = 1 - \cos A$, we have $\cos A = \frac{3}{5}$; whence, we take the side adjacent to the angle A as 3, and the hypothenuse as 5.

Then, $BC = \sqrt{AB^2 - AC^2} = \sqrt{25 - 9} = 4$.

Hence,

$$\sin A = \frac{4}{5} \qquad \cos A = \frac{3}{5}$$

$$\tan A = \frac{4}{3} \qquad \cot A = \frac{3}{4}$$

$$\sec A = \frac{5}{3} \qquad \operatorname{cosec} A = \frac{5}{4}$$

$$\operatorname{covers} A = 1 - \frac{4}{5} = \frac{1}{5} \qquad \operatorname{suvers} A = 1 + \frac{3}{5} = \frac{8}{5}$$

EXAMPLES.

In each case find the values of the remaining functions:

3. $\tan A = \frac{2}{3}$.

4. $\operatorname{covers} A = \frac{3}{5}$.

5. $\operatorname{cosec} A = 4$.

6. $\operatorname{vers} A = \frac{1}{4}$.

7. $\sin A = \frac{x}{y}$.

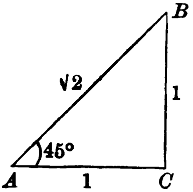
8. $\operatorname{suvers} A = \frac{3}{2}$.

9. $\cos A = \frac{1}{\sqrt{2}}$.

10. $\cot A = x$.

11. $\sec A = \frac{\sqrt{a^2 + b^2}}{b}$.

17. *To find the values of the functions of* 45°.



Let ABC be an isosceles right triangle, AC and BC being each equal to unity. Then $AB = \sqrt{AC^2 + BC^2} = \sqrt{1+1} = \sqrt{2}$. Also, the angles A and B are equal; and as their sum is 90°, each equals 45°.

Hence, by the definitions,

$\sin 45° = \frac{1}{\sqrt{2}} = \frac{1}{2}\sqrt{2}$ $\qquad$ $\cos 45° = \frac{1}{\sqrt{2}} = \frac{1}{2}\sqrt{2}$

$\tan 45° = 1$ $\qquad$ $\cot 45° = 1$

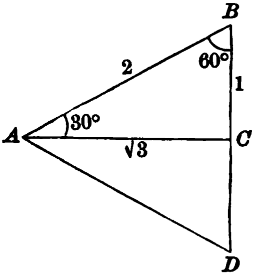
$\sec 45° = \sqrt{2}$ $\qquad$ $\text{cosec}\, 45° = \sqrt{2}$

$\text{vers}\, 45° = 1 - \frac{1}{\sqrt{2}} = \frac{\sqrt{2}-1}{\sqrt{2}}$ $\qquad$ $\text{covers}\, 45° = 1 - \frac{1}{\sqrt{2}} = \frac{\sqrt{2}-1}{\sqrt{2}}$

$$\text{suvers}\, 45° = 1 + \frac{1}{\sqrt{2}} = \frac{\sqrt{2}+1}{\sqrt{2}}$$

The second column might have been derived from the first by the principles of Art. 13, since 45° is the complement of itself.

18. *To find the values of the functions of* 30° *and* 60°.



Let ABD be an equilateral triangle having each side equal to 2. Draw AC perpendicular to BD, and, by Geometry, we have $BC = \frac{1}{2} BD = 1$, and $BAC = \frac{1}{2} BAD = 30°$. Then $AC = \sqrt{AB^2 - BC^2} = \sqrt{4-1} = \sqrt{3}$.

Then, by the definitions, from the triangle ABC, we have

$\sin 30° = \frac{1}{2} = \cos 60°$ $\qquad$ $\cos 30° = \frac{\sqrt{3}}{2} = \sin 60°$

$\tan 30° = \frac{1}{\sqrt{3}} = \frac{1}{3}\sqrt{3} = \cot 60°$ $\qquad$ $\cot 30° = \sqrt{3} = \tan 60°$

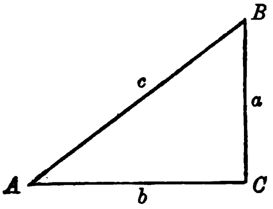
$$\sec 30^\circ = \frac{2}{\sqrt{3}} = \frac{2}{3}\sqrt{3} = \operatorname{cosec} 60^\circ \qquad \operatorname{cosec} 30^\circ = 2 = \sec 60^\circ$$

$$\operatorname{vers} 30^\circ = 1 - \frac{\sqrt{3}}{2} = \operatorname{covers} 60^\circ \qquad \operatorname{covers} 30^\circ = \frac{1}{2} = \operatorname{vers} 60^\circ$$

$$\operatorname{suvers} 30^\circ = 1 + \frac{\sqrt{3}}{2} \qquad \operatorname{suvers} 60^\circ = \frac{3}{2}$$

Or the functions of 60° may be derived from those of 30°, by Art. 13, since 60° is the complement of 30°.

FUNDAMENTAL THEOREMS.



19. *To prove the formula*

$$\sin^2 A + \cos^2 A = 1,$$

when A is any acute angle.

Note. Sin$^2 A$ denotes $(\sin A)^2$; that is, the square of the sine of A.

By Geometry, $$a^2 + b^2 = c^2$$

Dividing by c^2, $$\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1$$

Whence, by definition, $$(\sin A)^2 + (\cos A)^2 = 1$$

That is, $$\sin^2 A + \cos^2 A = 1 \qquad (5)$$

20. *To prove the formulæ*

$$\tan A = \frac{\sin A}{\cos A}, \qquad \cot A = \frac{\cos A}{\sin A}.$$

By definition, $$\tan A = \frac{a}{b} = \frac{\frac{a}{c}}{\frac{b}{c}} = \frac{\sin A}{\cos A} \qquad (6)$$

And since $$\cot A = \frac{1}{\tan A} \text{ (Art. 14)},$$

we have $$\cot A = \frac{\cos A}{\sin A} \qquad (7)$$

21. *To prove the formulæ*

$$\sec^2 A = 1 + \tan^2 A, \qquad \text{cosec}^2 A = 1 + \cot^2 A.$$

By Geometry, $$c^2 = a^2 + b^2$$

Dividing by b^2, $$\frac{c^2}{b^2} = 1 + \frac{a^2}{b^2}$$

That is, $$\sec^2 A = 1 + \tan^2 A \qquad (8)$$

Dividing by a^2, $$\frac{c^2}{a^2} = 1 + \frac{b^2}{a^2}$$

That is, $$\text{cosec}^2 A = 1 + \cot^2 A \qquad (9)$$

22. The following table expresses the values of each of the six principal functions in terms of the other five:

sin	...	$\sqrt{1-\cos^2}$	$\frac{\tan}{\sqrt{1+\tan^2}}$	$\frac{1}{\sqrt{1+\cot^2}}$	$\frac{\sqrt{\sec^2-1}}{\sec}$	$\frac{1}{\text{cosec}}$
cos	$\sqrt{1-\sin^2}$	...	$\frac{1}{\sqrt{1+\tan^2}}$	$\frac{\cot}{\sqrt{1+\cot^2}}$	$\frac{1}{\sec}$	$\frac{\sqrt{\text{cosec}^2-1}}{\text{cosec}}$
tan	$\frac{\sin}{\sqrt{1-\sin^2}}$	$\frac{\sqrt{1-\cos^2}}{\cos}$	...	$\frac{1}{\cot}$	$\sqrt{\sec^2-1}$	$\frac{1}{\sqrt{\text{cosec}^2-1}}$
cot	$\frac{\sqrt{1-\sin^2}}{\sin}$	$\frac{\cos}{\sqrt{1-\cos^2}}$	$\frac{1}{\tan}$	...	$\frac{1}{\sqrt{\sec^2-1}}$	$\sqrt{\text{cosec}^2-1}$
sec	$\frac{1}{\sqrt{1-\sin^2}}$	$\frac{1}{\cos}$	$\sqrt{1+\tan^2}$	$\frac{\sqrt{1+\cot^2}}{\cot}$	...	$\frac{\text{cosec}}{\sqrt{\text{cosec}^2-1}}$
cosec	$\frac{1}{\sin}$	$\frac{1}{\sqrt{1-\cos^2}}$	$\frac{\sqrt{1+\tan^2}}{\tan}$	$\sqrt{1+\cot^2}$	$\frac{\sec}{\sqrt{\sec^2-1}}$	...

Six of these values, that is, the reciprocal forms, were proved in Art. 14. The others may be derived by aid of the last three articles, and they are proposed as exercises for the student. We will work one as an illustration.

To prove the formula $\sin A = \frac{\tan A}{\sqrt{1+\tan^2 A}}$.

By Art. 20, $\tan A = \frac{\sin A}{\cos A} = \sin A \sec A$ (Art. 14).

But by Art. 21, $\sec A = \sqrt{1+\tan^2 A}$.

Substituting, $\tan A = \sin A \sqrt{1+\tan^2 A}$

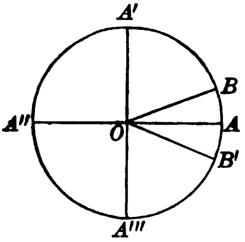
Hence, $\sin A = \frac{\tan A}{\sqrt{1+\tan^2 A}}$.

The formulæ of this article may be used advantageously in the solution of examples like those of Art. 16.

III. APPLICATION OF ALGEBRAIC SIGNS.

TRIGONOMETRICAL FUNCTIONS OF ANGLES IN GENERAL.

23. In Geometry, we are, as a rule, concerned with angles less than two right angles; but in Trigonometry it is convenient to regard them as unrestricted in magnitude.



In the circle $AA'A''A'''$, let AA'' and $A'A'''$ be a pair of diameters drawn at right angles to each other. Suppose a radius OB to start from the position OA, and revolve about the centre O as a pivot in the direction of OA'. When it coincides with OA', it has described an angular magnitude of 90°; when it coincides with OA'', of 180°; with OA''', of 270°; with OA, its starting point, of 360°; with OA' again, of 450°; etc. We thus see that a significance may be attached to any positive angular magnitude.

24. The interpretation of an angle, as the measure of the amount of rotation of a moving radius, enables us to distinguish between positive and negative angles. Thus, as a positive angle is taken to indicate revolution from the position OA in the direction of OA', a negative angle would naturally be taken as signifying revolution from the position OA in the *opposite* direction, towards OA'''.

To illustrate, if the angles AOB and AOB' are each equal to 30° in absolute magnitude, we should say that $AOB = +30°$, and $AOB' = -30°$.

We may thus conceive of *negative* angles of any magnitude whatever. It is immaterial which direction we consider the positive direction of rotation; but, having adopted a certain direction as positive, our subsequent operations must be in accordance.

25. The fixed line OA, from which the rotation is supposed to commence, is called the *initial line;* the rotating radius is called the *terminal line.* Either of the two lines forming an angle may be adopted as the initial line, the other being then the terminal line; thus, in the angle AOB, we may consider OA the initial line, and OB the terminal line, in which case the angle is positive; or, we may consider OB the initial line and OA the terminal line, in which case the angle is negative.

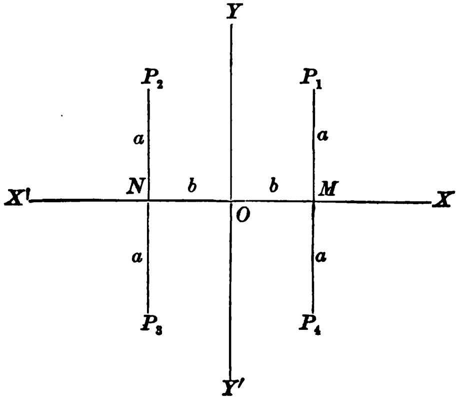
In designating an angle, we shall always write first the letter at the extremity of the initial line; thus, in designating the angle AOB, if we regard OA as the initial line, we should call it AOB, but if we regard OB as the initial line, we should call it BOA.

26. There are always two angles, one positive and the other negative, formed by a given initial and terminal line. Thus, there are formed by OA and OB' the positive angle AOB' greater than three right angles, and the negative angle AOB' less than one right angle. We shall distinguish between such angles by referring to them as "the positive angle AOB'," and "the negative angle AOB'," respectively.

It is evident that the terminal lines of two angles, which differ by a multiple of 360°, are coincident; thus, the angles 30°, 390°, 750°, −330°, −690°, etc., have the same terminal line.

27. The definitions of the six principal trigonometrical functions given in Arts. 9 and 10, being referred to a right triangle, apply of course to *acute* angles only. We must adopt more extended definitions in order to include *all* angles of any magnitude whatever, positive or negative.

RECTANGULAR CO-ORDINATES.



28. Let XX' and YY' be a pair of straight lines at right angles to each other. Let P_1 be any point in their plane, and draw P_1M perpendicular to XX'. The position of P_1 with reference to the lines XX' and YY' is known when the distances OM and P_1M are known.

These distances, OM and P_1M, are called the *rectangular co-ordinates* of the point P_1; OM is called the *abscissa*, and P_1M, the *ordinate*. The lines of reference, XX' and YY', are called the *axes of abscissas and ordinates*, or the *axes of X and Y*, respectively. Their intersection O is called the *origin*.

29. It is customary to denote the abscissa and ordinate of a point by the letters x and y, respectively. Thus, the fact that the abscissa of a point is equal to b and its ordinate to a is expressed by saying that for the point in question $x = b$ and $y = a$.

The same fact may be expressed still more concisely by referring to the point as the point (b, a); where the first term in the parenthesis is understood to refer to the abscissa, and the second term to the ordinate.

If a point lies upon the axis of X, its ordinate is evidently equal to zero; and the same is true of the abscissa of a point upon the axis of Y.

30. If, in the figure of Art. 28, $OM = ON = b$, and the perpendiculars $P_1M = P_2N = P_3N = P_4M = a$, the points P_1, P_2, P_3, and P_4 will have the same co-ordinates.

To avoid this ambiguity, the following conventions have been adopted: abscissas measured to the *right* of O are considered *positive*, and to the *left*, *negative*; and ordinates measured *above* the line XX' are considered *positive*, and *below*, *negative*.

Thus the co-ordinates of the four points will be:

Point.	*Abscissa.*	*Ordinate.*
P_1	b	a
P_2	$-b$	a
P_3	$-b$	$-a$
P_4	b	$-a$

Note. In the figures of this chapter, unless the contrary is stated, the small letters denote the *lengths* of the lines, without regard to their algebraic sign.

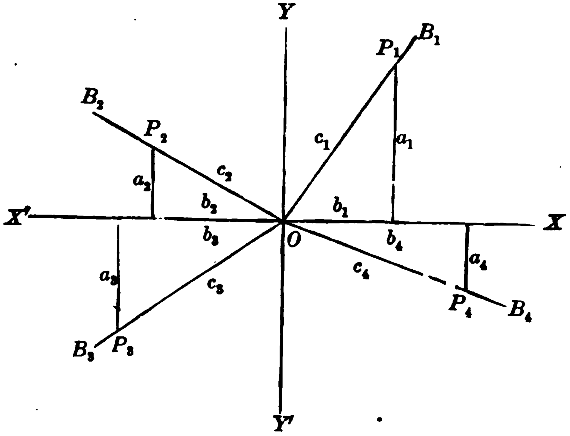
31. We may now give general definitions for the trigonometrical functions, applicable to all angles.

Let the initial line be taken as the axis of X, the vertex being the origin. Let any point be taken in the terminal line, and its rectangular co-ordinates constructed by dropping a perpendicular on to the initial line, produced if necessary. For convenience of designation we shall refer to that portion of the terminal line between the assumed point and the origin as the "distance" of the point. Then,

The SINE *is the ratio of the* ORDINATE *to the* DISTANCE.
The COSINE *is the ratio of the* ABSCISSA *to the* DISTANCE.
The TANGENT *is the ratio of the* ORDINATE *to the* ABSCISSA.
The COTANGENT *is the ratio of the* ABSCISSA *to the* ORDINATE.
The SECANT *is the ratio of the* DISTANCE *to the* ABSCISSA.
The COSECANT *is the ratio of the* DISTANCE *to the* ORDINATE.

32. The above definitions are seen to include those given for acute angles in Arts. 9 and 10. The definitions of the versed sine, coversed sine, and suversed sine, given in Art. 10, are sufficiently general to apply to all angles.

33. Let us now apply the definitions of Art. 31 to the angles XOB_1, XOB_2, XOB_3, and XOB_4 in the following figure:



Let P_1, P_2, P_3, and P_4 be any points in OB_1, OB_2, OB_3, OB_4, respectively; and let their rectangular co-ordinates be (b_1, a_1), $(-b_2, a_2)$, $(-b_3, -a_3)$, $(b_4, -a_4)$. Let the distance $OP_1 = c_1$, $OP_2 = c_2$, $OP_3 = c_3$, and $OP_4 = c_4$.

Then, by the definitions, we have

$$\begin{aligned}
\sin XOB_1 &= \frac{a_1}{c_1} & \cos XOB_1 &= \frac{b_1}{c_1} \\
\tan XOB_1 &= \frac{a_1}{b_1} & \cot XOB_1 &= \frac{b_1}{a_1} \\
\sec XOB_1 &= \frac{c_1}{b_1} & \operatorname{cosec} XOB_1 &= \frac{c_1}{a_1}
\end{aligned}$$

$$\begin{aligned}
\sin XOB_2 &= \frac{a_2}{c_2} & \cos XOB_2 &= \frac{-b_2}{c_2} = -\frac{b_2}{c_2} \\
\tan XOB_2 &= \frac{a_2}{-b_2} = -\frac{a_2}{b_2} & \cot XOB_2 &= \frac{-b_2}{a_2} = -\frac{b_2}{a_2} \\
\sec XOB_2 &= \frac{c_2}{-b_2} = -\frac{c_2}{b_2} & \operatorname{cosec} XOB_2 &= \frac{c_2}{a_2}
\end{aligned}$$

$$\begin{aligned}
\sin XOB_3 &= \frac{-a_3}{c_3} = -\frac{a_3}{c_3} & \cos XOB_3 &= \frac{-b_3}{c_3} = -\frac{b_3}{c_3} \\
\tan XOB_3 &= \frac{-a_3}{-b_3} = \frac{a_3}{b_3} & \cot XOB_3 &= \frac{-b_3}{-a_3} = \frac{b_3}{a_3} \\
\sec XOB_3 &= \frac{c_3}{-b_3} = -\frac{c_3}{b_3} & \operatorname{cosec} XOB_3 &= \frac{c_3}{-a_3} = -\frac{c_3}{a_3}
\end{aligned}$$

$$\begin{aligned}
\sin XOB_4 &= \frac{-a_4}{c_4} = -\frac{a_4}{c_4} & \cos XOB_4 &= \frac{b_4}{c_4} \\
\tan XOB_4 &= \frac{-a_4}{b_4} = -\frac{a_4}{b_4} & \cot XOB_4 &= \frac{b_4}{-a_4} = -\frac{b_4}{a_4} \\
\sec XOB_4 &= \frac{c_4}{b_4} & \operatorname{cosec} XOB_4 &= \frac{c_4}{-a_4} = -\frac{c_4}{a_4}
\end{aligned}$$

Since the terminal lines of two angles which differ by a multiple of 360° are coincident (Art. 26), it is evident from the definitions that the trigonometrical functions of two such angles are identical. Thus, the trigonometrical functions of the angles 50°, 410°, 770°, −310°, −670°, etc., are identical.

It is customary to express this fact by saying that the trigonometrical functions are *periodic*.

Again, any multiple of 360° may be added to or subtracted from an angle without altering its functions; and hence, any function of any angle may be expressed as the same function of a positive angle less than 360°. To illustrate,

$$\sin 830° = \sin(830° - 720°) = \sin 110°;$$
$$\cos(-140°) = \cos(-140° + 360°) = \cos 220°.$$

34. When the terminal line of an angle lies between OX and OY, we call the angle in the *first quadrant;* when between OY and OX', in the *second quadrant;* between OX' and OY', in the *third quadrant;* between OY' and OX, in the *fourth quadrant.*

To illustrate, any positive angle between 0° and 90°, or between 360° and 450°, etc., or any negative angle between −270° and −360°, or between −630° and −720°, etc., is in the first quadrant. Any positive angle between 90° and 180°, or between 450° and 540°, etc., or any negative angle between −180° and −270°, or between −540° and −630°, etc., is in the second quadrant; and so on.

In general terms, if m is 0 or any positive or negative integer, and A is an acute angle, the angles (see Art. 6)

$$4m\frac{\pi}{2} + A, \qquad (4m+1)\frac{\pi}{2} + A, \qquad (4m+2)\frac{\pi}{2} + A,$$
$$\text{and} \qquad (4m+3)\frac{\pi}{2} + A,$$

are in the first, second, third, and fourth quadrants, respectively.

35. We observe, by inspection of the results in Art. 33, the following important points in relation to the algebraic signs of the trigonometrical functions in the different quadrants:

For any angle in the first quadrant, all the functions are positive.

In the second quadrant, the sine and cosecant are positive, and the cosine, tangent, cotangent, and secant are negative.

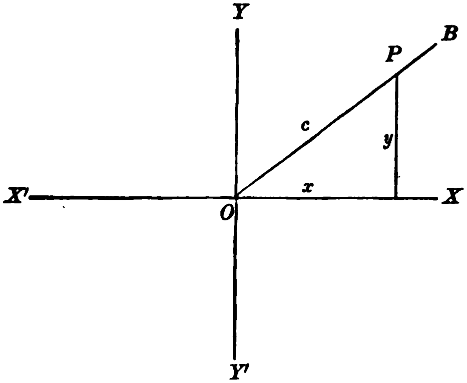
In the third quadrant, the tangent and cotangent are positive, and the sine, cosine, secant, and cosecant are negative.

In the fourth quadrant, the cosine and secant are positive, and the sine, tangent, cotangent, and cosecant are negative.

It is customary to express the foregoing principles in a tabular form, as follows:

Function.	*First Quad.*	*Second Quad.*	*Third Quad.*	*Fourth Quad.*
Sine and cosecant	+	+	−	−
Cosine and secant	+	−	−	+
Tangent and cotangent. . .	+	−	+	−

36. *To find the values of the functions of* 0°, 90°, 180°, *and* 270°.



Let XOB be any angle, and let P be any point in OB. Let the co-ordinates of P be (x, y), and let $OP = c$, and $XOB = A$. Then, by the definitions,

$$\sin A = \frac{y}{c} \qquad \tan A = \frac{y}{x} \qquad \sec A = \frac{c}{x}$$

$$\cos A = \frac{x}{c} \qquad \cot A = \frac{x}{y} \qquad \operatorname{cosec} A = \frac{c}{y}$$

Suppose the line OB to start from the position OX, and to revolve about O as a pivot in the direction of OY, the point P remaining at the constant distance c from O.

When OB coincides with OX, $A = 0°$; and as P is then a point on OX at the distance c from O, its co-ordinates are $(c, 0)$. Thus, $x = c$, and $y = 0$.

Hence, the equations become

$$\sin 0° = \frac{0}{c} = 0 \qquad \tan 0° = \frac{0}{c} = 0 \qquad \sec 0° = \frac{c}{c} = 1$$

$$\cos 0° = \frac{c}{c} = 1 \qquad \cot 0° = \frac{c}{0} = \infty \qquad \operatorname{cosec} 0° = \frac{c}{0} = \infty$$

When OB coincides with OY, $A = 90°$, $x = 0$, $y = c$. Hence,

$$\sin 90° = \frac{c}{c} = 1 \qquad \tan 90° = \frac{c}{0} = \infty \qquad \sec 90° = \frac{c}{0} = \infty$$

$$\cos 90° = \frac{0}{c} = 0 \qquad \cot 90° = \frac{0}{c} = 0 \qquad \operatorname{cosec} 90° = \frac{c}{c} = 1$$

When OB coincides with OX', $A = 180°$, $x = -c$, $y = 0$. Hence,

$$\sin 180° = \frac{0}{c} = 0 \qquad \cos 180° = \frac{-c}{c} = -1$$

$$\tan 180° = \frac{0}{-c} = 0 \qquad \cot 180° = \frac{-c}{0} = \infty$$

$$\sec 180° = \frac{c}{-c} = -1 \qquad \operatorname{cosec} 180° = \frac{c}{0} = \infty$$

When OB coincides with OY', $A = 270°$, $x = 0$, and $y = -c$. Hence,

$$\sin 270° = \frac{-c}{c} = -1 \qquad \cos 270° = \frac{0}{c} = 0$$

$$\tan 270° = \frac{-c}{0} = \infty \qquad \cot 270° = \frac{0}{-c} = 0$$

$$\sec 270° = \frac{c}{0} = \infty \qquad \operatorname{cosec} 270° = \frac{c}{-c} = -1$$

Continuing the revolution, we should find for the functions of 360° the same values as for the functions of 0°, for 450° the same as for 90°, etc. Also, we should find for the functions of −90°, −180°, −270°, −360°, −450°, etc., the same values as for 270°, 180°, 90°, 0°, 270°, etc., respectively.

Note. The equation $\cot 0° = \infty$ cannot be taken literally; it merely signifies that as an angle decreases indefinitely its cotangent increases without limit, and when the angle becomes 0° the cotangent can no longer be expressed in finite terms. A similar interpretation is to be given to the equations $\operatorname{cosec} 0° = \infty$, $\tan 90° = \infty$, etc.

The foregoing results may be expressed in a tabular form as follows:

Angle.	*Sin.*	*Cos.*	*Tan.*	*Cot.*	*Sec.*	*Cosec.*
0°	0	1	0	∞	1	∞
90°	1	0	∞	0	∞	1
180°	0	−1	0	∞	−1	∞
270°	−1	0	∞	0	∞	−1
360°	0	1	0	∞	1	∞

37. Expressing the results of the preceding article in general terms, if n is 0, or any positive or negative integer, we shall have the functions of

$$4n\frac{\pi}{2}, \quad (4n+1)\frac{\pi}{2}, \quad (4n+2)\frac{\pi}{2}, \text{ and } (4n+3)\frac{\pi}{2},$$

the same as the functions of 0°, 90°, 180°, and 270°, respectively.

This may be put in a tabular form as follows:

Angle.	*Sin.*	*Cos.*	*Tan.*	*Cot.*	*Sec.*	*Cosec.*
$4n\frac{\pi}{2}$	0	1	0	∞	1	∞
$(4n+1)\frac{\pi}{2}$	1	0	∞	0	∞	1
$(4n+2)\frac{\pi}{2}$	0	-1	0	∞	-1	∞
$(4n+3)\frac{\pi}{2}$	-1	0	∞	0	∞	-1

38. *To trace the changes in the values of the six principal functions as the angle varies from* 0° *to* 360°.

With the figure and notation of Art. 36, it is evident that as the angle varies from 0° to 360°, x commences with the value c, and gradually decreases until the angle equals 90°, in which case it equals 0. It now becomes negative, and increases numerically until the angle equals 180°, in which case it equals $-c$; then decreases numerically until the angle equals 270°, in which case it equals 0. From 270° to 360° it is positive, and increases from 0 to c.

Also, y commences with the value 0, gradually increases until the angle equals 90°, in which case it equals c; then decreases until the angle equals 180°, in which case it equals 0. It now becomes negative, and increases numerically until the angle equals 270°, in which case it equals $-c$; and, finally, decreases numerically until the angle equals 360°, in which case it equals 0.

Then, since the sine of the angle is $\frac{y}{c}$, it is evident that, as the angle varies from 0° to 90°, the sine increases from 0 to 1; from 90° to 180° it decreases from 1 to 0; from 180° to 270° it is negative, and decreases from 0 to -1; and from 270° to 360° it increases from -1 to 0.

Since the cosine is $\frac{x}{c}$, as the angle varies from 0° to 90°, the cosine decreases from 1 to 0; from 90° to 180° it is negative, and decreases from 0 to -1; from 180° to 270° it is negative, and increases from -1 to 0; and from 270° to 360° it increases from 0 to 1.

Since the tangent is $\frac{y}{x}$, from 0° to 90° it increases from 0 to ∞; from 90° to 180° it is negative, and increases from $-\infty$ to 0; from 180° to 270° it increases from 0 to ∞; and from 270° to 360° it is negative, and increases from $-\infty$ to 0.

The values of the remaining functions change as follows, as the angle varies from 0° to 360°:

From 0° to 90°,

the cotangent decreases from ∞ to 0;
the secant increases from 1 to ∞;
the cosecant decreases from ∞ to 1.

From 90° to 180°,

the cotangent decreases from 0 to $-\infty$;
the secant increases from $-\infty$ to -1;
the cosecant increases from 1 to ∞.

From 180° to 270°,

the cotangent decreases from ∞ to 0;
the secant decreases from -1 to $-\infty$;
the cosecant increases from $-\infty$ to -1.

From 270° to 360°,

the cotangent decreases from 0 to $-\infty$;
the secant decreases from ∞ to 1;
the cosecant decreases from -1 to $-\infty$.

Note. In the foregoing results, wherever the symbol ∞ appears, it must be interpreted according to the note of Art. 36. For example, the statement that, as the angle varies from 0° to 90° its tangent increases from 0 to ∞, simply means that the tangent increases without limit.

39. It is evident from the preceding article that the values of the sine and cosine of an angle must lie between 1 and -1; of the tangent and cotangent, between ∞ and $-\infty$; and of the secant and cosecant, between 1 and ∞, or between -1 and $-\infty$.

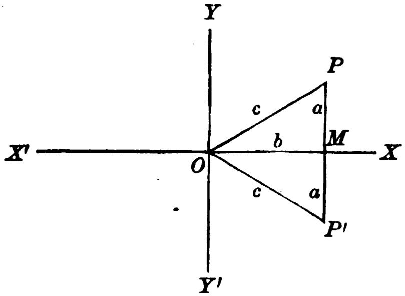
40. *To find the values of the functions of* $(-A)$ *in terms of those of* A.

There will be four cases to consider, according as A is in the first, second, third, or fourth quadrant.

In each figure we suppose the positive angle XOP to represent the angle A, and the negative angle XOP' the angle $(-A)$. Let PP' be drawn perpendicular to XX'; then, since the right triangles OPM and $OP'M$ have the hypothenuse and an acute angle of the one equal respectively to the hypothenuse and an acute angle of the other, they are equal, and $OP = OP'$ and $PM = P'M$.

Let $PM = P'M = a$, $OM = b$, and $OP = OP' = c$.

Case I. *When* A *is in the first quadrant.*



By the definitions,

$$\sin A = \frac{a}{c} \qquad \tan A = \frac{a}{b} \qquad \sec A = \frac{c}{b}$$

$$\cos A = \frac{b}{c} \qquad \cot A = \frac{b}{a} \qquad \operatorname{cosec} A = \frac{c}{a}$$

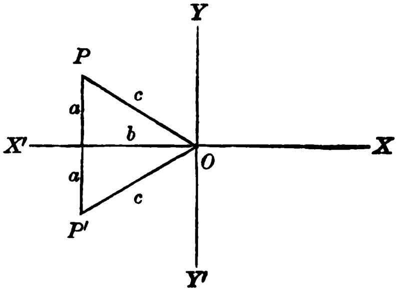
$$\sin(-A) = -\frac{a}{c} \qquad \tan(-A) = -\frac{a}{b} \qquad \sec(-A) = \frac{c}{b}$$

$$\cos(-A) = \frac{b}{c} \qquad \cot(-A) = -\frac{b}{a} \qquad \operatorname{cosec}(-A) = -\frac{c}{a}$$

Whence,

$$\left.\begin{array}{ll} \sin(-A) = -\sin A & \cos(-A) = \cos A \\ \tan(-A) = -\tan A & \cot(-A) = -\cot A \\ \sec(-A) = \sec A & \operatorname{cosec}(-A) = -\operatorname{cosec} A \end{array}\right\} \quad (10)$$

CASE II. *When A is in the second quadrant.*



We now have

$$\sin A = \frac{a}{c} \qquad \tan A = -\frac{a}{b} \qquad \sec A = -\frac{c}{b}$$

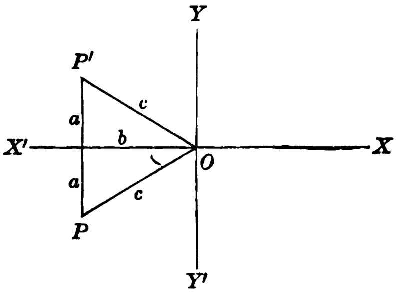
$$\cos A = -\frac{b}{c} \qquad \cot A = -\frac{b}{a} \qquad \operatorname{cosec} A = \frac{c}{a}$$

$$\sin(-A) = -\frac{a}{c} \qquad \tan(-A) = \frac{a}{b} \qquad \sec(-A) = -\frac{c}{b}$$

$$\cos(-A) = -\frac{b}{c} \qquad \cot(-A) = \frac{b}{a} \qquad \operatorname{cosec}(-A) = -\frac{c}{a}$$

Whence we obtain the equations (10) as before.

Case III. *When A is in the third quadrant.*



We now have

$$\sin A = -\frac{a}{c} \qquad \tan A = \frac{a}{b} \qquad \sec A = -\frac{c}{b}$$

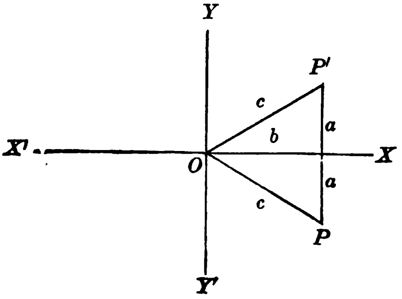
$$\cos A = -\frac{b}{c} \qquad \cot A = \frac{b}{a} \qquad \operatorname{cosec} A = -\frac{c}{a}$$

$$\sin(-A) = \frac{a}{c} \qquad \tan(-A) = -\frac{a}{b} \qquad \sec(-A) = -\frac{c}{b}$$

$$\cos(-A) = -\frac{b}{c} \qquad \cot(-A) = -\frac{b}{a} \qquad \operatorname{cosec}(-A) = \frac{c}{a}$$

Giving the equations (10) as before.

Case IV. *When A is in the fourth quadrant.*



We now have

$$\sin A = -\frac{a}{c} \qquad \tan A = -\frac{a}{b} \qquad \sec A = \frac{c}{b}$$

$$\cos A = \frac{b}{c} \qquad \cot A = -\frac{b}{a} \qquad \operatorname{cosec} A = -\frac{c}{a}$$

$$\sin(-A) = \frac{a}{c} \qquad \tan(-A) = \frac{a}{b} \qquad \sec(-A) = \frac{c}{b}$$

$$\cos(-A) = \frac{b}{c} \qquad \cot(-A) = \frac{b}{a} \qquad \operatorname{cosec}(-A) = \frac{c}{a}$$

Giving the equations (**10**) as before.

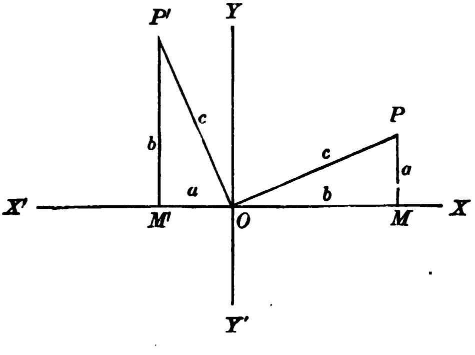
41. *To find the values of the functions of* $90° + A$ *in terms of those of* A.

Here, also, we may distinguish four cases. In each figure we suppose the positive angle XOP to represent the angle A, and the positive angle XOP' the angle $90° + A$. The distances OP and OP' are taken equal, and PM and $P'M'$ are drawn perpendicular to XX'.

Since OP' is perpendicular to OP, and OM' to PM, the angles $P'OM'$ and OPM are equal by Geometry; and hence the right triangles OPM and $OP'M'$ are equal, and $P'M' = OM$ and $OM' = PM$.

Let $PM = OM' = a$, $OM = P'M' = b$, and $OP = OP' = c$.

CASE I. *When A is in the first quadrant.*



We now have

$$\sin A = \frac{a}{c} \qquad \tan A = \frac{a}{b} \qquad \sec A = \frac{c}{b}$$

$$\cos A = \frac{b}{c} \qquad \cot A = \frac{b}{a} \qquad \operatorname{cosec} A = \frac{c}{a}$$

$$\sin(90° + A) = \frac{b}{c} \qquad \cos(90° + A) = -\frac{a}{c}$$

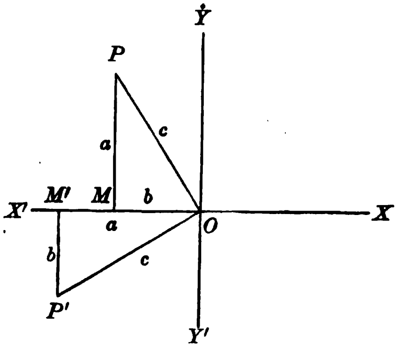
$$\tan(90° + A) = -\frac{b}{a} \qquad \cot(90° + A) = -\frac{a}{b}$$

$$\sec(90° + A) = -\frac{c}{a} \qquad \operatorname{cosec}(90° + A) = \frac{c}{b}$$

Whence,

$$\left.\begin{aligned} \sin(90° + A) &= \cos A & \cos(90° + A) &= -\sin A \\ \tan(90° + A) &= -\cot A & \cot(90° + A) &= -\tan A \\ \sec(90° + A) &= -\operatorname{cosec} A & \operatorname{cosec}(90° + A) &= \sec A \end{aligned}\right\}(11)$$

Case II. *When A is in the second quadrant.*



We now have

$$\sin A = \frac{a}{c} \qquad \tan A = -\frac{a}{b} \qquad \sec A = -\frac{c}{b}$$

$$\cos A = -\frac{b}{c} \qquad \cot A = -\frac{b}{a} \qquad \operatorname{cosec} A = \frac{c}{a}$$

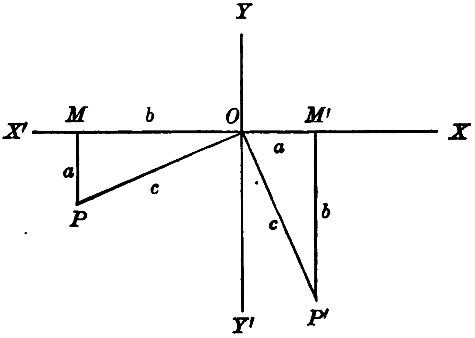
$$\sin(90^\circ+A) = -\frac{b}{c} \qquad \cos(90^\circ+A) = -\frac{a}{c}$$

$$\tan(90^\circ+A) = \frac{b}{a} \qquad \cot(90^\circ+A) = \frac{a}{b}$$

$$\sec(90^\circ+A) = -\frac{c}{a} \qquad \operatorname{cosec}(90^\circ+A) = -\frac{c}{b}$$

Giving the equations (**11**) as before.

Case III. *When A is in the third quadrant.*



We now have

$$\sin A = -\frac{a}{c} \qquad \tan A = \frac{a}{b} \qquad \sec A = -\frac{c}{b}$$

$$\cos A = -\frac{b}{c} \qquad \cot A = \frac{b}{a} \qquad \operatorname{cosec} A = -\frac{c}{a}$$

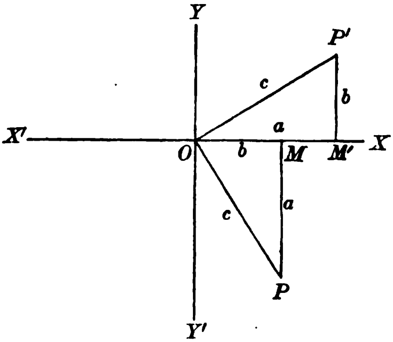
$$\sin(90^\circ+A) = -\frac{b}{c} \qquad \cos(90^\circ+A) = \frac{a}{c}$$

$$\tan(90^\circ+A) = -\frac{b}{a} \qquad \cot(90^\circ+A) = -\frac{a}{b}$$

$$\sec(90^\circ+A) = \frac{c}{a} \qquad \operatorname{cosec}(90^\circ+A) = -\frac{c}{b}$$

Giving the equations (**11**) as before.

CASE IV. *When A is in the fourth quadrant.*



We now have

$$\sin A = -\frac{a}{c} \qquad \tan A = -\frac{a}{b} \qquad \sec A = \frac{c}{b}$$

$$\cos A = \frac{b}{c} \qquad \cot A = -\frac{b}{a} \qquad \operatorname{cosec} A = -\frac{c}{a}$$

$$\sin(90^\circ + A) = \frac{b}{c} \qquad \cos(90^\circ + A) = \frac{a}{c}$$

$$\tan(90^\circ + A) = \frac{b}{a} \qquad \cot(90^\circ + A) = \frac{a}{b}$$

$$\sec(90^\circ + A) = \frac{c}{a} \qquad \operatorname{cosec}(90^\circ + A) = \frac{c}{b}$$

Giving the equations **(11)** as before.

42. *To find the values of the functions of* $90^\circ - A$ *in terms of those of* A.

Putting $-A$ for A in **(11)**, Art. 41, we have

$$\sin(90^\circ - A) = \cos(-A) \qquad \cos(90^\circ - A) = -\sin(-A)$$

$$\tan(90^\circ - A) = -\cot(-A) \qquad \cot(90^\circ - A) = -\tan(-A)$$

$$\sec(90^\circ - A) = -\operatorname{cosec}(-A) \qquad \operatorname{cosec}(90^\circ - A) = \sec(-A)$$

Whence, by Art. 40,

$$\sin(90^\circ - A) = \cos A \qquad \cos(90^\circ - A) = \sin A$$

$$\tan(90^\circ - A) = \cot A \qquad \cot(90^\circ - A) = \tan A$$

$$\sec(90^\circ - A) = \operatorname{cosec} A \qquad \operatorname{cosec}(90^\circ - A) = \sec A$$

We have here the proof for all angles of the formulæ for acute angles which were proved in Art. 13.

43. *To find the values of the functions of* $180° - A$ *in terms of those of* A.

Putting $90° - A$ for A in (**11**), Art. 41, we have

$$\begin{aligned} \sin(180° - A) &= \cos(90° - A) \\ \tan(180° - A) &= -\cot(90° - A) \\ \sec(180° - A) &= -\operatorname{cosec}(90° - A) \end{aligned}$$

$$\begin{aligned} \cos(180° - A) &= -\sin(90° - A) \\ \cot(180° - A) &= -\tan(90° - A) \\ \operatorname{cosec}(180° - A) &= \sec(90° - A) \end{aligned}$$

Whence, by Art. 42,

$$\begin{aligned} \sin(180° - A) &= \sin A & \qquad \cos(180° - A) &= -\cos A \\ \tan(180° - A) &= -\tan A & \qquad \cot(180° - A) &= -\cot A \\ \sec(180° - A) &= -\sec A & \qquad \operatorname{cosec}(180° - A) &= \operatorname{cosec} A \end{aligned}$$

These equations express the values of the functions of the supplement of an angle in terms of the functions of the angle itself.

44. *To find the values of the functions of* $180° + A$ *in terms of those of* A.

Putting $90° + A$ for A in (**11**), Art. 41, we have

$$\begin{aligned} \sin(180° + A) &= \cos(90° + A) \\ \tan(180° + A) &= -\cot(90° + A) \\ \sec(180° + A) &= -\operatorname{cosec}(90° + A) \end{aligned}$$

$$\begin{aligned} \cos(180° + A) &= -\sin(90° + A) \\ \cot(180° + A) &= -\tan(90° + A) \\ \operatorname{cosec}(180° + A) &= \sec(90° + A) \end{aligned}$$

Whence, by (11), Art. 41,

$$\begin{aligned} \sin(180^\circ+A) &= -\sin A & \qquad \cos(180^\circ+A) &= -\cos A \\ \tan(180^\circ+A) &= \tan A & \qquad \cot(180^\circ+A) &= \cot A \\ \sec(180^\circ+A) &= -\sec A & \qquad \operatorname{cosec}(180^\circ+A) &= -\operatorname{cosec} A \end{aligned}$$

45. *To find the values of the functions of* $270^\circ-A$ *in terms of those of* A.

Putting $180^\circ-A$ for A in (11), Art. 41, we have

$$\begin{aligned} \sin(270^\circ-A) &= \cos(180^\circ-A) \\ \tan(270^\circ-A) &= -\cot(180^\circ-A) \\ \sec(270^\circ-A) &= -\operatorname{cosec}(180^\circ-A) \end{aligned}$$

$$\begin{aligned} \cos(270^\circ-A) &= -\sin(180^\circ-A) \\ \cot(270^\circ-A) &= -\tan(180^\circ-A) \\ \operatorname{cosec}(270^\circ-A) &= \sec(180^\circ-A) \end{aligned}$$

Whence, by Art. 43,

$$\begin{aligned} \sin(270^\circ-A) &= -\cos A & \qquad \cos(270^\circ-A) &= -\sin A \\ \tan(270^\circ-A) &= \cot A & \qquad \cot(270^\circ-A) &= \tan A \\ \sec(270^\circ-A) &= -\operatorname{cosec} A & \qquad \operatorname{cosec}(270^\circ-A) &= -\sec A \end{aligned}$$

46. *To find the values of the functions of* $270^\circ+A$ *in terms of those of* A.

Putting $180^\circ+A$ for A in (11), Art. 41, we have

$$\begin{aligned} \sin(270^\circ+A) &= \cos(180^\circ+A) \\ \tan(270^\circ+A) &= -\cot(180^\circ+A) \\ \sec(270^\circ+A) &= -\operatorname{cosec}(180^\circ+A) \end{aligned}$$

$$\begin{aligned} \cos(270^\circ+A) &= -\sin(180^\circ+A) \\ \cot(270^\circ+A) &= -\tan(180^\circ+A) \\ \operatorname{cosec}(270^\circ+A) &= \sec(180^\circ+A) \end{aligned}$$

Whence, by Art. 44,

$\sin(270°+A) = -\cos A$ $\quad\quad$ $\cos(270°+A) = \sin A$

$\tan(270°+A) = -\cot A$ $\quad\quad$ $\cot(270°+A) = -\tan A$

$\sec(270°+A) = \operatorname{cosec} A$ $\quad\quad$ $\operatorname{cosec}(270°+A) = -\sec A$

47. In a similar manner may be proved the two following groups, which are left as exercises for the student:

$\sin(360°-A) = -\sin A$ $\quad\quad$ $\cos(360°-A) = \cos A$

$\tan(360°-A) = -\tan A$ $\quad\quad$ $\cot(360°-A) = -\cot A$

$\sec(360°-A) = \sec A$ $\quad\quad$ $\operatorname{cosec}(360°-A) = -\operatorname{cosec} A$

$\sin(360°+A) = \sin A$ $\quad\quad$ $\cos(360°+A) = \cos A$

$\tan(360°+A) = \tan A$ $\quad\quad$ $\cot(360°+A) = \cot A$

$\sec(360°+A) = \sec A$ $\quad\quad$ $\operatorname{cosec}(360°+A) = \operatorname{cosec} A$

We may see the truth of the last two groups in another way; for the functions of the angles $360°-A$ and $360°+A$ will be the same as the functions of the angles $-A$ and A, respectively, in accordance with the general principle of Art. 33, that the functions of two angles which differ by a multiple of 360° are identical.

48. The formulæ of Arts. 42 to 47 may also be proved after the manner of Arts. 40 and 41; that is, by drawing a figure and considering A at first in the first quadrant, then in the second, and so on. The proofs of the formulæ by this second method are proposed as exercises for the student.

49. The operations illustrated in Arts. 42 to 47 may obviously be extended indefinitely; we should find for the angles $450°-A$, $810°-A$, or, in general, for any angle in the form $(4n+1)\frac{\pi}{2}-A$, where n is 0, or any positive or negative integer, the same results as for $90°-A$; for $540°-A$, $900°-A$, or any angle in the form $(4n+2)\frac{\pi}{2}-A$, the same results as for $180°-A$; and so on.

50. The results of the preceding article may also be expressed in the following rule, which is derived by inspection from the formulæ of Arts. 40 to 47.

Any function of 0°, *or an* **even** *multiple of* 90°, *plus or minus* A, *is the* **same** *function of* A; *and any function of an* **odd** *multiple of* 90°, *plus or minus* A, *is the* **complementary** *function of* A.

For the algebraic sign, regard A *as acute, and apply the table of the signs of the functions in the different quadrants as given in* Art. 35.

1. Find the value of $\sec(990°-A)$.

Since 990° is an odd multiple of 90°, by the rule the absolute value of the result is $\operatorname{cosec} A$. And if A is acute, $990°-A$ is in the third quadrant, in which the sign of the secant is negative.

Hence, $\sec(990°-A)=-\operatorname{cosec} A$. *Ans.*

2. Find the value of $\tan(-180°+A)$.

Since 180° is an even multiple of 90°, the absolute value of the result is $\tan A$. And if A is acute, $-180°+A$ is in the third quadrant, in which the sign of the tangent is positive.

Hence, $\tan(-180°+A)=\tan A$. *Ans.*

EXAMPLES.

Find the values of the following:

3. $\cot(540°-A)$.
4. $\sin(-270°+A)$.
5. $\sec(450°+A)$.
6. $\tan(1260°-A)$.
7. $\cot(-90°-A)$.
8. $\operatorname{cosec}(-360°+A)$.
9. $\cos(270°-A)$.
10. $\operatorname{cosec}(720°+A)$.
11. $\sec(630°+A)$.
12. $\sin(-180°-A)$.
13. $\tan(0°-A)$.
14. $\cos(1080°-A)$.

The functions of any positive or negative angle whatever may be expressed in terms of the functions of a certain acute angle.

15. Find the value of $\sin 317°$ in terms of the functions of an acute angle.

We have, by the rule,

$$\sin 317° = \sin(270° + 47°) = -\cos 47°;$$

or,

$$= \sin(360° - 43°) = -\sin 43°.$$

Express the following in terms of the functions of acute angles:

16. $\cos 152° 45'$. **18.** $\sec(-72°)$. **20.** $\cot(-120°)$.

17. $\tan 522°$. **19.** $\operatorname{cosec} 230°$. **21.** $\sin 865° 12'$.

Express the following in terms of the functions of acute angles less than 45°:

22. $\cot 163° 31'$. **23.** $\sin 588° 20'$. **24.** $\sec(-307°)$.

51. The following may be derived from the formulæ of Arts. 40 to 47, using the results of Arts. 17 and 18. They are proposed as exercises for the student.

Angle.	*Sin.*	*Cos.*	*Tan.*	*Cot.*	*Sec.*	*Cosec.*
120°	$\frac{1}{2}\sqrt{3}$	$-\frac{1}{2}$	$-\sqrt{3}$	$-\frac{1}{3}\sqrt{3}$	-2	$\frac{2}{3}\sqrt{3}$
135°	$\frac{1}{2}\sqrt{2}$	$-\frac{1}{2}\sqrt{2}$	-1	-1	$-\sqrt{2}$	$\sqrt{2}$
150°	$\frac{1}{2}$	$-\frac{1}{2}\sqrt{3}$	$-\frac{1}{3}\sqrt{3}$	$-\sqrt{3}$	$-\frac{2}{3}\sqrt{3}$	2
210°	$-\frac{1}{2}$	$-\frac{1}{2}\sqrt{3}$	$\frac{1}{3}\sqrt{3}$	$\sqrt{3}$	$-\frac{2}{3}\sqrt{3}$	-2
225°	$-\frac{1}{2}\sqrt{2}$	$-\frac{1}{2}\sqrt{2}$	1	1	$-\sqrt{2}$	$-\sqrt{2}$
240°	$-\frac{1}{2}\sqrt{3}$	$-\frac{1}{2}$	$\sqrt{3}$	$\frac{1}{3}\sqrt{3}$	-2	$-\frac{2}{3}\sqrt{3}$
300°	$-\frac{1}{2}\sqrt{3}$	$\frac{1}{2}$	$-\sqrt{3}$	$-\frac{1}{3}\sqrt{3}$	2	$-\frac{2}{3}\sqrt{3}$
315°	$-\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{2}$	-1	-1	$\sqrt{2}$	$-\sqrt{2}$
330°	$-\frac{1}{2}$	$\frac{1}{2}\sqrt{3}$	$-\frac{1}{3}\sqrt{3}$	$-\sqrt{3}$	$\frac{2}{3}\sqrt{3}$	-2

EXERCISES.

Find the values of the following:

1. $\cos 495°$.	3. $\operatorname{cosec}(-240°)$.	5. $\sin 510°$.
2. $\sec 870°$.	4. $\tan 585°$.	6. $\cot(-120°)$.

52. We have seen in the preceding articles that the angles $180°-A$, $360°+A$, $540°-A$, etc., or $-180°-A$, $-360°+A$, $-540°-A$, etc., have their sines equal to $\sin A$; or, in general, that every angle in the form $n\pi+(-1)^n A$, where n is 0, or any positive or negative integer, has its sine equal to $\sin A$.

It follows, then, that if we have given a value of $\sin A$, and are required to find the angle corresponding, there will be an infinite number of solutions, each of which will be in the form $n\pi+(-1)^n A$.

To illustrate, let $\sin A=\frac{1}{2}$.

We then know, from Art. 18, that one value of A is 30° or $\frac{\pi}{6}$. Then all the solutions are obtained by giving to n the values 0, or any positive or negative integer in the expression $n\pi+(-1)^n\frac{\pi}{6}$.

Thus, if $n=1$, one solution is $\pi-\frac{\pi}{6}$ or $\frac{5\pi}{6}$;

if $n=2$, another is $2\pi+\frac{\pi}{6}$ or $\frac{13\pi}{6}$; etc.

The expression $n\pi+(-1)^n\frac{\pi}{6}$ is called the *general value* of A in the equation $\sin A=\frac{1}{2}$. The general value in any case may be obtained by finding any one of the values of A, and substituting it in the general expression $n\pi+(-1)^n A$.

In a similar manner we may derive the following:

All angles corresponding to a given value of $\cos A$ or of $\sec A$ are given by $2n\pi \pm A$.

All angles corresponding to a given value of $\tan A$ or of $\cot A$ are given by $n\pi + A$.

All angles corresponding to a given value of $\operatorname{cosec} A$ are given by $n\pi + (-1)^n A$.

EXAMPLES.

1. Find the general value of A when $\sec A = -2$.

By Art. 51 we see that one value of A is 120° or $\frac{2\pi}{3}$. Hence, the general value is $2n\pi \pm \frac{2\pi}{3}$.

2. Find the general value of A when $\tan A = \sqrt{3}$ and $\cos A = -\frac{1}{2}$.

By Art. 18, if $\tan A = \sqrt{3}$, one value of A is 60° or $\frac{\pi}{3}$, and hence, the general value corresponding to the first condition is $n\pi + \frac{\pi}{3}$.

And if $\cos A = -\frac{1}{2}$, by Art. 51, one value of A is 120° or $\frac{2\pi}{3}$, and the general value corresponding to this condition is $2n\pi \pm \frac{2\pi}{3}$.

Now, the expressions $n\pi + \frac{\pi}{3}$ and $2n\pi \pm \frac{2\pi}{3}$ denote the *same series* of angles only when n is odd in the first expression, and the lower sign is taken in the second. Hence, the general value of A satisfying both conditions is either

$$(2m+1)\pi + \frac{\pi}{3} \text{ or } 2n\pi - \frac{2\pi}{3}.$$

In each of the following find the general value of A:

3. $\operatorname{cosec} A = -\frac{2}{3}\sqrt{3}$.

4. $\tan A = 1$.

5. $\sin A = \frac{1}{2}\sqrt{2}$.

6. $\cot A = -\sqrt{3}$.

7. $\cos A = -\frac{1}{2}$.

8. $\sec A = \sqrt{2}$.

9. $\tan A = -1$ and $\sec A = -\sqrt{2}$.

10. $\sin A = -\frac{1}{2}\sqrt{3}$ and $\cot A = \frac{1}{3}\sqrt{3}$.

11. $\cos A = \frac{1}{2}$ and $\cot A = \frac{1}{3}\sqrt{3}$.

12. $\operatorname{cosec} A = -\sqrt{2}$ and $\tan A = -1$.

13. Find the general value of A when $\cot A = -\tan 20°$.

Since $-\tan 20° = \cot(90° + 20°) = \cot\frac{11\pi}{18}$, one value of A is $\frac{11\pi}{18}$. Hence, the general value of A is $n\pi + \frac{11\pi}{18}$.

In the following find the general values of A:

14. $\sin A = \cos 40°$.

15. $\sec A = -\operatorname{cosec}\frac{3\pi}{5}$.

16. $\tan A = \cot 117°$.

17. $\cos A = -\cos\frac{\pi}{8}$.

PROOFS OF THE FUNDAMENTAL FORMULÆ FOR ANY ANGLE.

53. It follows immediately from the general definitions of Art. 31, exactly as in Art. 14, that

$$\sin A = \frac{1}{\operatorname{cosec} A} \qquad \tan A = \frac{1}{\cot A} \qquad \sec A = \frac{1}{\cos A}$$

$$\cos A = \frac{1}{\sec A} \qquad \cot A = \frac{1}{\tan A} \qquad \operatorname{cosec} A = \frac{1}{\sin A}$$

These formulæ are consequently proved universally.

54. *To prove the formulæ*

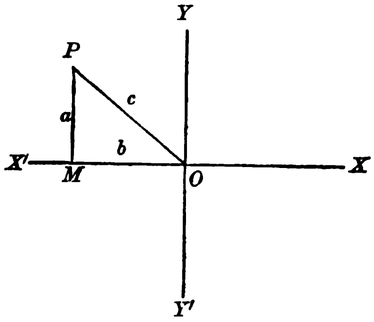
$$\tan A = \frac{\sin A}{\cos A} \quad \textit{and} \quad \cot A = \frac{\cos A}{\sin A}$$

when A is any angle.

In Art. 20 we proved the formulæ for any acute angle; and the method of proof employed evidently holds for any angle in the first quadrant.

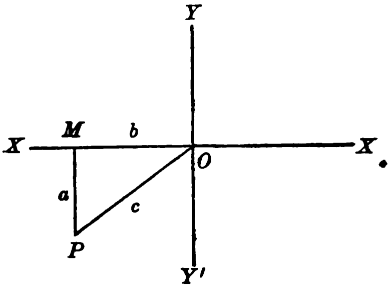
In the following figures let XOP represent the angle A, and let PM be drawn perpendicular to XX'. Let $PM = a$, $OM = b$, and $OP = c$.

CASE II. *When A is in the second quadrant.*



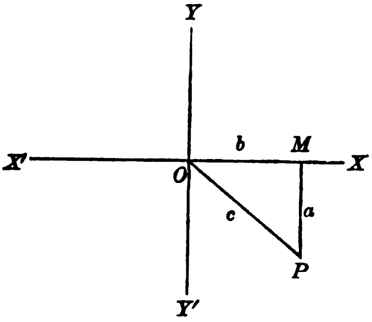
By the definitions, $\tan A = -\frac{a}{b} = \dfrac{\frac{a}{c}}{-\frac{b}{c}} = \dfrac{\sin A}{\cos A}$

CASE III. *When A is in the third quadrant.*



Here, $\tan A = \frac{a}{b} = \dfrac{-\frac{a}{c}}{-\frac{b}{c}} = \dfrac{\sin A}{\cos A}$

Case IV. *When A is in the fourth quadrant.*



Here, $$\tan A = -\frac{a}{b} = \frac{-\frac{a}{c}}{\frac{b}{c}} = \frac{\sin A}{\cos A}.$$

The formula $\tan A = \frac{\sin A}{\cos A}$ has thus been shown to hold universally; and, since $\cot A = \frac{1}{\tan A}$ for any angle whatever (Art. 53), we have $\cot A = \frac{\cos A}{\sin A}$ for any value of A.

55. *To prove the formulæ*

$\sin^2 A + \cos^2 A = 1$, $\sec^2 A = 1 + \tan^2 A$, $\operatorname{cosec}^2 A = 1 + \cot^2 A$,

for any value of A.

In Arts. 19 and 21 these formulæ were proved for any acute angle; or, what is the same thing, for any angle in the first quadrant.

If A is in the second, third, or fourth quadrant, we have, from the figures of the preceding article,

$$a^2 + b^2 = c^2,$$

which may be written in the forms

$$\frac{a^2}{c^2} + \frac{b^2}{c^2} = 1, \quad \frac{c^2}{b^2} = 1 + \frac{a^2}{b^2}, \quad \text{or} \quad \frac{c^2}{a^2} = 1 + \frac{b^2}{a^2}.$$

But in either of the figures, since the square of a quantity is always positive, we have

$$\sin^2 A = \frac{a^2}{c^2} \qquad \tan^2 A = \frac{a^2}{b^2} \qquad \sec^2 A = \frac{c^2}{b^2}$$

$$\cos^2 A = \frac{b^2}{c^2} \qquad \cot^2 A = \frac{b^2}{a^2} \qquad \operatorname{cosec}^2 A = \frac{c^2}{a^2}$$

Hence, we have universally

$$\sin^2 A + \cos^2 A = 1, \quad \sec^2 A = 1 + \tan^2 A, \quad \operatorname{cosec}^2 A = 1 + \cot^2 A.$$

56. *Given the value of one of the functions of an angle, to find the values of the remaining functions.* (Compare Art. 16.)

1. Given $\sin A = -\frac{2}{3}$; find the values of the remaining functions.

By Art. 53, $\operatorname{cosec} A = \frac{1}{\sin A} = -\frac{3}{2}$

By Art. 55, $\cos^2 A = 1 - \sin^2 A = 1 - \frac{4}{9} = \frac{5}{9}$

Or, $\cos A = \pm \frac{1}{3}\sqrt{5}$.

Taking $\cos A = \frac{1}{3}\sqrt{5}$, we have

$$\tan A = \frac{\sin A}{\cos A} = \frac{-\frac{2}{3}}{\frac{1}{3}\sqrt{5}} = -\tfrac{2}{5}\sqrt{5}$$

$$\cot A = \frac{\cos A}{\sin A} = \frac{\frac{1}{3}\sqrt{5}}{-\frac{2}{3}} = -\tfrac{1}{2}\sqrt{5}$$

$$\sec A = \frac{1}{\cos A} = \frac{1}{\frac{1}{3}\sqrt{5}} = \tfrac{3}{5}\sqrt{5}$$

Taking $\cos A = -\frac{1}{3}\sqrt{5}$, we have, in a similar manner,

$$\tan A = \tfrac{2}{5}\sqrt{5}$$

$$\cot A = \tfrac{1}{2}\sqrt{5}$$

$$\sec A = -\tfrac{3}{5}\sqrt{5}$$

Thus the two solutions to the example are,

$$\cos A = \pm \tfrac{1}{3}\sqrt{5}, \quad \tan A = \mp \tfrac{2}{5}\sqrt{5}, \quad \cot A = \mp \tfrac{1}{2}\sqrt{5},$$
$$\sec A = \pm \tfrac{3}{5}\sqrt{5}, \quad \operatorname{cosec} A = -\frac{3}{2},$$

where the upper signs and the lower signs must be taken together.

In examples like the above the remaining functions will, in general, have two values each, except the reciprocal function, which will have but one value; and there are usually two solutions. The reason for this ambiguity is that in general two angles correspond to a given value of a function, as is evident from the definitions of Art. 31.

Thus, in the last example, the point of reference may be regarded as having its ordinate equal to -2, and its distance equal to 3; and there are obviously two such points; one in the third, the other in the fourth quadrant.

2. Given $\cot A = 2$; find the values of the remaining functions.

By Art. 53, $\tan A = \dfrac{1}{\cot A} = \dfrac{1}{2}$

By Art. 55, $\operatorname{cosec}^2 A = 1 + \cot^2 A = 1 + 4 = 5$

Or, $\operatorname{cosec} A = \pm\sqrt{5}$

Then, $\sin A = \dfrac{1}{\operatorname{cosec} A} = \pm\dfrac{1}{\sqrt{5}} = \pm\tfrac{1}{5}\sqrt{5}$

By Art. 54, $\cos A = \sin A \cot A = \pm\tfrac{2}{5}\sqrt{5}$

Whence, $\sec A = \dfrac{1}{\cos A} = \pm\dfrac{1}{\frac{2}{5}\sqrt{5}} = \pm\tfrac{1}{2}\sqrt{5}$

Thus the two solutions are,

$$\sin A = \pm\tfrac{1}{5}\sqrt{5}, \quad \cos A = \pm\tfrac{2}{5}\sqrt{5}, \quad \tan A = \frac{1}{2},$$
$$\sec A = \pm\tfrac{1}{2}\sqrt{5}, \quad \operatorname{cosec} A = \pm\sqrt{5},$$

where the upper signs and the lower signs must be taken together.

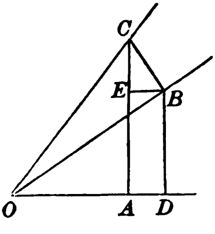
EXAMPLES.

In each case find the values of the remaining functions:

3. $\sin A = \frac{1}{4}$.	**7.** $\cos A = -\frac{5}{13}$.	**11.** $\cot A = \frac{1}{x}$.
4. $\operatorname{cosec} A = -2$.	**8.** $\cot A = \frac{3}{2}$.	**12.** $\sin A = -\frac{2}{a-1}$.
5. $\tan A = -1$.	**9.** $\operatorname{cosec} A = x$.	**13.** $\sec A = \frac{\sqrt{a^2+b^2}}{b}$.
6. $\sec A = \frac{5}{3}$.	**10.** $\cos A = -\frac{a}{b}$.	**14.** $\tan A = \frac{x+1}{x}$.

IV. GENERAL FORMULÆ.

57. *To find the values of* $\sin(x+y)$ *and* $\cos(x+y)$ *in terms of the sines and cosines of* x *and* y.



Let AOB denote the angle x and BOC the angle y; then AOC will denote the angle $x+y$. From any point C in OC draw CA perpendicular to OA, and CB perpendicular to OB. Also draw BD perpendicular to OA and BE perpendicular to AC.

Since the sides of the angle BCE are respectively perpendicular to the sides of the angle AOB, by Geometry these angles are equal; that is, the angle $BCE = x$.

We then have

$$\sin(x+y)=\frac{AC}{OC}=\frac{BD+CE}{OC}=\frac{BD}{OC}+\frac{CE}{OC}$$

But $$\frac{BD}{OC}=\frac{BD}{OB}\times\frac{OB}{OC}=\sin x\cos y$$

and $$\frac{CE}{OC}=\frac{CE}{BC}\times\frac{BC}{OC}=\cos x\sin y$$

Hence, $$\sin(x+y)=\sin x\cos y+\cos x\sin y \qquad (12)$$

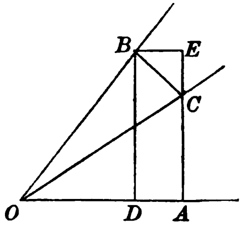
Again, $$\cos(x+y)=\frac{OA}{OC}=\frac{OD-BE}{OC}=\frac{OD}{OC}-\frac{BE}{OC}$$

But $$\frac{OD}{OC}=\frac{OD}{OB}\times\frac{OB}{OC}=\cos x\cos y$$

and $$\frac{BE}{OC}=\frac{BE}{BC}\times\frac{BC}{OC}=\sin x\sin y$$

Hence, $$\cos(x+y)=\cos x\cos y-\sin x\sin y \qquad (13)$$

58. *To find the values of* $\sin(x-y)$ *and* $\cos(x-y)$ *in terms of the sines and cosines of* x *and* y.



Let AOB denote the angle x, and BOC the angle y; then AOC will denote the angle $x-y$. From any point C in OC draw CA perpendicular to OA, and CB perpendicular to OB. Also draw BD perpendicular to OA, and BE perpendicular to AC produced.

Since the sides of the angle BCE are respectively perpendicular to the sides of the angle AOB, by Geometry these angles are equal; that is, the angle $BCE=x$.

We now have

$$\sin(x-y)=\frac{AC}{OC}=\frac{BD-CE}{OC}=\frac{BD}{OC}-\frac{CE}{OC}$$

But $$\frac{BD}{OC}=\frac{BD}{OB}\times\frac{OB}{OC}=\sin x\cos y$$

and $$\frac{CE}{OC}=\frac{CE}{BC}\times\frac{BC}{OC}=\cos x\sin y$$

Hence, $$\sin(x-y)=\sin x\cos y-\cos x\sin y \qquad \textbf{(14)}$$

Again, $$\cos(x-y)=\frac{OA}{OC}=\frac{OD+BE}{OC}=\frac{OD}{OC}+\frac{BE}{OC}$$

But $$\frac{OD}{OC}=\frac{OD}{OB}\times\frac{OB}{OC}=\cos x\cos y$$

and $$\frac{BE}{OC}=\frac{BE}{BC}\times\frac{BC}{OC}=\sin x\sin y$$

Hence, $$\cos(x-y)=\cos x\cos y+\sin x\sin y \qquad \textbf{(15)}$$

59. The fundamental formulæ, (**12**), (**13**), (**14**), and (**15**), are of great importance, since a large number of other formulæ can be derived from them; and it is necessary to show that they hold universally.

It is obvious that the proof of Art. 57 is not general, since in the figure we have taken x and y as acute angles, and $x+y<90°$. Also, in the proof of Art. 58, we have taken x and y as acute angles, and $x>y$.

To prove the formulæ universally, we will first show that (**14**) and (**15**) are true for all values of x and y; and from this result we can deduce the general proof of (**12**) and (**13**), as in Art. 62.

60. We must first prove (**14**) and (**15**) when x and y are acute, and $x<y$.

Since $y>x$, we have, by Art. 58,

$$\sin(y-x)=\sin y\cos x-\cos y\sin x$$
$$\cos(y-x)=\cos y\cos x+\sin y\sin x$$

But, by Art. 40, putting $x-y$ for $-A$, whence $A=y-x$, we have

$$\sin(x-y)=-\sin(y-x), \text{ and } \cos(x-y)=\cos(y-x)$$

Hence,
$$\sin(x-y)=\sin x\cos y-\cos x\sin y$$
$$\cos(x-y)=\cos x\cos y+\sin x\sin y$$

We have thus proved the formulæ when x and y are any two acute angles; or, what is the same thing, when they are any two angles in the first quadrant. If they are not both in the first quadrant, there will be fifteen other cases to consider.

CASE II. *When x is in the first quadrant and y in the second.*

Let $y=90°+y'$, where y' is in the first quadrant.

Then,

$$\begin{aligned}\sin(x-y)&=\sin(x-y'-90°)=-\cos(x-y') &&\text{(Art. 50)}\\ &=-\cos x\cos y'-\sin x\sin y' &&\text{(Art. 58)}\\ &=-\cos x\sin(90°+y')+\sin x\cos(90°+y')\\ &=\sin x\cos y-\cos x\sin y\end{aligned}$$

And,

$$\begin{aligned}\cos(x-y)&=\cos(x-y'-90°)=\sin(x-y')\\ &=\sin x\cos y'-\cos x\sin y'\\ &=\sin x\sin(90°+y')+\cos x\cos(90°+y')\\ &=\cos x\cos y+\sin x\sin y\end{aligned}$$

We may prove the remaining cases in a similar manner: for Case III., we should take x in the first quadrant and y in the third; for Case IV., x in the first and y in the fourth; and so on. The proofs of these other fourteen cases are left as exercises for the student.

The proof of any one of the cases may also be obtained by drawing a figure similar to that in Art. 58; and the student is recommended to work out a few results by this method.

Another general proof of (**14**) and (**15**) will be found in the following article.

61. By Art. 41, if a and b are any assigned values of x and y, for which the formulæ (**14**) and (**15**) are true,

$$\left.\begin{aligned}\sin\{90^\circ+(a-b)\} &= \cos(a-b)\\ &= \cos a\cos b+\sin a\sin b\\ \cos\{90^\circ+(a-b)\} &= -\sin(a-b)\\ &= -\sin a\cos b+\cos a\sin b\end{aligned}\right\}\ (\text{A})$$

But, by Arts. 41 and 50,

$$\begin{aligned}\sin(90^\circ+a) &= \cos a, & \cos(90^\circ+a) &= -\sin a\\ \sin(b-90^\circ) &= -\cos b, & \cos(b-90^\circ) &= \sin b\end{aligned}$$

Hence, the first of the equations (A) may be written in either of the forms

$$\sin\{(90^\circ+a)-b\}=\sin(90^\circ+a)\cos b-\cos(90^\circ+a)\sin b;$$

or, $\sin\{a-(b-90^\circ)\}=\sin a\cos(b-90^\circ)-\cos a\sin(b-90^\circ)$.

And the second may be written in either of the forms

$$\cos\{(90^\circ+a)-b\}=\cos(90^\circ+a)\cos b+\sin(90^\circ+a)\sin b;$$

or, $\cos\{a-(b-90^\circ)\}=\cos a\cos(b-90^\circ)+\sin a\sin(b-90^\circ)$,

and we observe that the last four equations are in accordance with (**14**) and (**15**).

That is, if the formulæ (**14**) and (**15**) are true for any assigned values of x and y, such as a and b, they are also true when a is increased, or b diminished, by 90°.

Again, by Art. 50,

$$\left.\begin{aligned}\sin\{-90^\circ+(a-b)\} &= -\cos(a-b)\\ &= -\cos a\cos b-\sin a\sin b\\ \cos\{-90^\circ+(a-b)\} &= \sin(a-b)\\ &= \sin a\cos b-\cos a\sin b\end{aligned}\right\}\ (\text{B})$$

Since, by Arts. 41 and 50,

$$\begin{aligned}\sin(a-90^\circ) &= -\cos a & \cos(a-90^\circ) &= \sin a\\ \sin(90^\circ+b) &= \cos b & \cos(90^\circ+b) &= -\sin b\end{aligned}$$

the first of the equations (**B**) may be written in either of the forms

$$\sin\{(a-90^\circ)-b\}=\sin(a-90^\circ)\cos b-\cos(a-90^\circ)\sin b;$$

or, $$\sin\{a-(b+90^\circ)\}=\sin a\cos(b+90^\circ)-\cos a\sin(b+90^\circ),$$

and the second of (**B**) in either of the forms

$$\cos\{(a-90^\circ)-b\}=\cos(a-90^\circ)\cos b+\sin(a-90^\circ)\sin b,$$
$$\cos\{a-(b+90^\circ)\}=\cos a\cos(b+90^\circ)+\sin a\sin(b+90^\circ),$$

all of which are in accordance with (**14**) and (**15**).

That is, if (**14**) and (**15**) are true for any assigned values of x and y, such as a and b, they are also true when a is diminished, or b increased, by 90°.

But we have proved them to be true for any acute values of x and y; consequently they are universally true.

62. The general proof of (**12**) and (**13**) may now be given.

In Arts. 58 and 60, (**14**) and (**15**) were proved for any values of x and y; putting $-y$ in place of y, these become

$$\sin(x+y)=\sin x\cos(-y)-\cos x\sin(-y)$$
$$\cos(x+y)=\cos x\cos(-y)+\sin x\sin(-y)$$

Reducing by aid of Art. 40,

$$\sin(x+y)=\sin x\cos y+\cos x\sin y$$
$$\cos(x+y)=\cos x\cos y-\sin x\sin y$$

63. Dividing (**12**) by (**13**), we obtain, by Art. 20,

$$\tan(x+y)=\frac{\sin x\cos y+\cos x\sin y}{\cos x\cos y-\sin x\sin y}$$

Dividing each term of the fraction by $\cos x\cos y$, we have

$$\tan(x+y)=\frac{\dfrac{\sin x\cos y}{\cos x\cos y}+\dfrac{\cos x\sin y}{\cos x\cos y}}{\dfrac{\cos x\cos y}{\cos x\cos y}-\dfrac{\sin x\sin y}{\cos x\cos y}}$$

$$=\frac{\tan x+\tan y}{1-\tan x\tan y} \qquad (16)$$

In a similar manner may be derived

$$\tan(x-y)=\frac{\tan x-\tan y}{1+\tan x\tan y} \qquad (17)$$

64. Dividing (13) by (12), we obtain

$$\cot(x+y)=\frac{\cos x\cos y-\sin x\sin y}{\sin x\cos y+\cos x\sin y}$$

Dividing each term of the fraction by $\sin x\sin y$, we have

$$\cot(x+y)=\frac{\dfrac{\cos x\cos y}{\sin x\sin y}-\dfrac{\sin x\sin y}{\sin x\sin y}}{\dfrac{\sin x\cos y}{\sin x\sin y}+\dfrac{\cos x\sin y}{\sin x\sin y}}$$

$$=\frac{\cot x\cot y-1}{\cot y+\cot x} \qquad (18)$$

And in a similar manner we derive

$$\cot(x-y)=\frac{\cot x\cot y+1}{\cot y-\cot x} \qquad (19)$$

65. The following formulæ may be deduced by methods similar to those employed in Arts. 63 and 64, and they are left as exercises for the student.

$$\frac{\sin(x+y)}{\sin(x-y)}=\frac{\tan x+\tan y}{\tan x-\tan y} \qquad (20)$$

$$\frac{\cos(x+y)}{\cos(x-y)}=\frac{1-\tan x\tan y}{1+\tan x\tan y} \qquad (21)$$

$$\frac{\sin(x+y)}{\cos(x-y)}=\frac{\tan x+\tan y}{1+\tan x\tan y} \qquad (22)$$

$$\frac{\sin(x-y)}{\cos(x+y)}=\frac{\tan x-\tan y}{1-\tan x\tan y} \qquad (23)$$

66. Multiplying (**12**) by (**14**), we obtain

$$\begin{aligned}\sin(x+y)\sin(x-y)&\\ &=\sin^2 x\cos^2 y-\cos^2 x\sin^2 y\\ &=\sin^2 x(1-\sin^2 y)-(1-\sin^2 x)\sin^2 y &&\text{(Art. 19)}\\ &=\sin^2 x-\sin^2 x\sin^2 y-\sin^2 y+\sin^2 x\sin^2 y\\ &=\sin^2 x-\sin^2 y &&\textbf{(24)}\end{aligned}$$

This result may also be written by Art. 19,

$$\begin{aligned}\sin(x+y)\sin(x-y)&=1-\cos^2 x-(1-\cos^2 y)\\ &=\cos^2 y-\cos^2 x &&\textbf{(25)}\end{aligned}$$

In a similar manner are derived the following,

$$\cos(x+y)\cos(x-y)=\cos^2 x-\sin^2 y=\cos^2 y-\sin^2 x \qquad \textbf{(26)}$$

67. Putting $y=x$ in (**12**), we have

$$\begin{aligned}\sin 2x&=\sin x\cos x+\cos x\sin x\\ &=2\sin x\cos x &&\textbf{(27)}\end{aligned}$$

Similarly, from (**13**), (**16**), and (**18**), we obtain

$$\cos 2x=\cos^2 x-\sin^2 x \qquad \textbf{(28)}$$

$$\tan 2x=\frac{2\tan x}{1-\tan^2 x} \qquad \textbf{(29)}$$

$$\cot 2x=\frac{\cot^2 x-1}{2\cot x} \qquad \textbf{(30)}$$

By Art. 19, (**28**) may be written in either of the forms

$$\cos 2x=2\cos^2 x-1 \qquad \textbf{(31)}$$

or

$$\cos 2x=1-2\sin^2 x \qquad \textbf{(32)}$$

68. From (**31**) and (**32**) we have

$$1+\cos 2x=2\cos^2 x \text{ and } 1-\cos 2x=2\sin^2 x \qquad \textbf{(33)}$$

Dividing the second of these by the first,

$$\frac{1-\cos 2x}{1+\cos 2x}=\frac{2\sin^2 x}{2\cos^2 x}=\tan^2 x \qquad \textbf{(34)}$$

Dividing (**27**) by (**33**), we have

$$\frac{\sin 2x}{1+\cos 2x}=\frac{2\sin x\cos x}{2\cos^2 x}=\tan x \tag{35}$$

and

$$\frac{\sin 2x}{1-\cos 2x}=\frac{2\sin x\cos x}{2\sin^2 x}=\cot x \tag{36}$$

69. In (**33**), (**34**), (**35**), and (**36**), writing x in place of $2x$, and therefore $\frac{1}{2}x$ in place of x, we obtain

$$2\cos^2\tfrac{1}{2}x=1+\cos x,\ 2\sin^2\tfrac{1}{2}x=1-\cos x \tag{37}$$

$$\tan^2\tfrac{1}{2}x=\frac{1-\cos x}{1+\cos x} \tag{38}$$

$$\tan\tfrac{1}{2}x=\frac{\sin x}{1+\cos x} \tag{39}$$

$$\cot\tfrac{1}{2}x=\frac{\sin x}{1-\cos x} \tag{40}$$

And, by inverting the last two,

$$\tan\tfrac{1}{2}x=\frac{1-\cos x}{\sin x} \tag{41}$$

$$\cot\tfrac{1}{2}x=\frac{1+\cos x}{\sin x} \tag{42}$$

70. By Art. 57,

$$\begin{aligned}\sin(x+y+z)&=\sin\{(x+y)+z\}\\&=\sin(x+y)\cos z+\cos(x+y)\sin z\\&=(\sin x\cos y+\cos x\sin y)\cos z\\&\qquad+(\cos x\cos y-\sin x\sin y)\sin z\\&=\sin x\cos y\cos z+\cos x\sin y\cos z\\&\qquad+\cos x\cos y\sin z-\sin x\sin y\sin z\end{aligned} \tag{43}$$

Similarly, from (**13**) we derive

$$\begin{aligned}\cos(x+y+z)&=\cos x\cos y\cos z-\sin x\sin y\cos z\\&\qquad-\sin x\cos y\sin z-\cos x\sin y\sin z\end{aligned} \tag{44}$$

From (**16**) we obtain

$$\tan(x+y+z) = \frac{\tan(x+y)+\tan z}{1-\tan(x+y)\tan z}$$

$$= \frac{\dfrac{\tan x+\tan y}{1-\tan x\tan y}+\tan z}{1-\tan z\left(\dfrac{\tan x+\tan y}{1-\tan x\tan y}\right)}$$

$$= \frac{\tan x+\tan y+\tan z-\tan x\tan y\tan z}{1-\tan y\tan z-\tan z\tan x-\tan x\tan y} \qquad (45)$$

In a similar manner we derive the functions of $x+y-z$, $x-y+z$, etc.; in fact, by successive applications of (**12**), (**13**), (**14**), and (**15**), we may obtain the functions of any compound angle whatever in terms of the functions of its component terms.

71. In (**12**), (**13**), and (**16**), put $y=2x$; we then have

$$\sin 3x = \sin x\cos 2x+\cos x\sin 2x$$

$$\cos 3x = \cos x\cos 2x-\sin x\sin 2x$$

$$\tan 3x = \frac{\tan x+\tan 2x}{1-\tan x\tan 2x}$$

Whence, by Art. 67, we obtain

$$\sin 3x = \sin x(1-2\sin^2 x)+\cos x(2\sin x\cos x)$$

$$= \sin x-2\sin^3 x+2\sin x(1-\sin^2 x) \qquad \text{(Art. 19)}$$

$$= 3\sin x-4\sin^3 x \qquad (46)$$

$$\cos 3x = \cos x(2\cos^2 x-1)-\sin x(2\sin x\cos x)$$

$$= 2\cos^3 x-\cos x-2\cos x(1-\cos^2 x) \qquad \text{(Art. 19)}$$

$$= 4\cos^3 x-3\cos x \qquad (47)$$

$$\tan 3x = \frac{\tan x+\dfrac{2\tan x}{1-\tan^2 x}}{1-\tan x\left(\dfrac{2\tan x}{1-\tan^2 x}\right)}$$

$$= \frac{3\tan x-\tan^3 x}{1-3\tan^2 x} \qquad (48)$$

These three results may also be obtained from (**43**), (**44**), and (**45**), by putting y and z each equal to x.

72. From (**12**), (**13**), (**14**), and (**15**), we obtain, by addition and subtraction,

$$\begin{aligned}\sin(x+y)+\sin(x-y)&=2\sin x\cos y\\ \sin(x+y)-\sin(x-y)&=2\cos x\sin y\\ \cos(x+y)+\cos(x-y)&=2\cos x\cos y\\ \cos(x+y)-\cos(x-y)&=-2\sin x\sin y\end{aligned}$$

Let $x+y=a$, and $x-y=b$.

Then, $x=\frac{1}{2}(a+b)$, and $y=\frac{1}{2}(a-b)$.

Substituting, we have

$$\begin{aligned}\sin a+\sin b&=2\sin\tfrac{1}{2}(a+b)\cos\tfrac{1}{2}(a-b)\\ \sin a-\sin b&=2\cos\tfrac{1}{2}(a+b)\sin\tfrac{1}{2}(a-b)\\ \cos a+\cos b&=2\cos\tfrac{1}{2}(a+b)\cos\tfrac{1}{2}(a-b)\\ \cos a-\cos b&=-2\sin\tfrac{1}{2}(a+b)\sin\tfrac{1}{2}(a-b)\end{aligned}$$

Replacing a and b by x and y, we obtain

$$\sin x+\sin y=2\sin\tfrac{1}{2}(x+y)\cos\tfrac{1}{2}(x-y) \qquad (\mathbf{49})$$

$$\sin x-\sin y=2\cos\tfrac{1}{2}(x+y)\sin\tfrac{1}{2}(x-y) \qquad (\mathbf{50})$$

$$\cos x+\cos y=2\cos\tfrac{1}{2}(x+y)\cos\tfrac{1}{2}(x-y) \qquad (\mathbf{51})$$

$$\cos x-\cos y=-2\sin\tfrac{1}{2}(x+y)\sin\tfrac{1}{2}(x-y) \qquad (\mathbf{52})$$

73. By dividing (**49**) by (**50**), we have

$$\begin{aligned}\frac{\sin x+\sin y}{\sin x-\sin y}&=\frac{2\sin\frac{1}{2}(x+y)\cos\frac{1}{2}(x-y)}{2\cos\frac{1}{2}(x+y)\sin\frac{1}{2}(x-y)}\\ &=\tan\tfrac{1}{2}(x+y)\cot\tfrac{1}{2}(x-y)\\ &=\frac{\tan\frac{1}{2}(x+y)}{\tan\frac{1}{2}(x-y)}\ (\text{Art. }14) \qquad (\mathbf{53})\end{aligned}$$

The student may prove the following in a similar manner:

$$\frac{\sin x+\sin y}{\cos x+\cos y}=\tan\tfrac{1}{2}(x+y) \qquad (\mathbf{54})$$

$$\frac{\sin x-\sin y}{\cos x+\cos y}=\tan\tfrac{1}{2}(x-y) \qquad (\mathbf{55})$$

$$\frac{\sin x+\sin y}{\cos x-\cos y}=-\cot\tfrac{1}{2}(x-y) \tag{56}$$

$$\frac{\sin x-\sin y}{\cos x-\cos y}=-\cot\tfrac{1}{2}(x+y) \tag{57}$$

$$\frac{\cos x+\cos y}{\cos x-\cos y}=-\cot\tfrac{1}{2}(x+y)\cot\tfrac{1}{2}(x-y) \tag{58}$$

74. *To find the values of the functions of* 15° *and* 75°.

In (**14**) and (**15**) put $x=45°$ and $y=30°$; then, by Arts. 17 and 18,

$$\sin 15°=\sin(45°-30°)=\sin 45°\cos 30°-\cos 45°\sin 30°$$
$$=\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}-\frac{\sqrt{2}}{2}\cdot\frac{1}{2}=\frac{\sqrt{6}-\sqrt{2}}{4}$$

$$\cos 15°=\cos(45°-30°)=\cos 45°\cos 30°+\sin 45°\sin 30°$$
$$=\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}+\frac{\sqrt{2}}{2}\cdot\frac{1}{2}=\frac{\sqrt{6}+\sqrt{2}}{4}$$

Putting $x=30°$ in (**41**) and (**42**), we have

$$\tan 15°=\frac{1-\cos 30°}{\sin 30°}=\frac{1-\frac{1}{2}\sqrt{3}}{\frac{1}{2}}=2-\sqrt{3}$$

$$\cot 15°=\frac{1+\cos 30°}{\sin 30°}=\frac{1+\frac{1}{2}\sqrt{3}}{\frac{1}{2}}=2+\sqrt{3}$$

Also,

$$\sec 15°=\frac{1}{\cos 15°}=\frac{4}{\sqrt{6}+\sqrt{2}}=\frac{4(\sqrt{6}-\sqrt{2})}{(\sqrt{6}+\sqrt{2})(\sqrt{6}-\sqrt{2})}$$
$$=\sqrt{6}-\sqrt{2}$$

$$\operatorname{cosec} 15°=\frac{1}{\sin 15°}=\frac{4}{\sqrt{6}-\sqrt{2}}=\sqrt{6}+\sqrt{2}$$

And, as 75° is the complement of 15°, we shall have

$$\sin 75°=\frac{\sqrt{6}+\sqrt{2}}{4} \qquad \cos 75°=\frac{\sqrt{6}-\sqrt{2}}{4}$$

$$\tan 75°=2+\sqrt{3} \qquad \cot 75°=2-\sqrt{3}$$

$$\sec 75°=\sqrt{6}+\sqrt{2} \qquad \operatorname{cosec} 75°=\sqrt{6}-\sqrt{2}$$

75. *To find the values of the functions of* $22\frac{1}{2}°$ *and* $67\frac{1}{2}°$.

In (**37**), put $x = 45°$. We then have, by Art. 17,

$$2\sin^2 22\tfrac{1}{2}° = 1 - \frac{\sqrt{2}}{2} \qquad 2\cos^2 22\tfrac{1}{2}° = 1 + \frac{\sqrt{2}}{2}$$

Dividing each equation by 2, and extracting the square root,

$$\sin 22\tfrac{1}{2}° = \frac{\sqrt{2-\sqrt{2}}}{2} \qquad \cos 22\tfrac{1}{2}° = \frac{\sqrt{2+\sqrt{2}}}{2}$$

The positive sign of the radical is taken here, because, since $22\frac{1}{2}°$ is an angle in the first quadrant, all its functions are positive.

Putting $x = 45°$ in (**41**) and (**42**), we have

$$\tan 22\tfrac{1}{2}° = \frac{1 - \cos 45°}{\sin 45°} = \frac{1 - \frac{1}{2}\sqrt{2}}{\frac{1}{2}\sqrt{2}} = \frac{2 - \sqrt{2}}{\sqrt{2}} = \sqrt{2} - 1$$

$$\cot 22\tfrac{1}{2}° = \frac{1 + \cos 45°}{\sin 45°} = \sqrt{2} + 1$$

Then, by Art. 22,

$$\sec 22\tfrac{1}{2}° = \sqrt{1 + \tan^2 22\tfrac{1}{2}°}$$
$$= \sqrt{1 + 2 - 2\sqrt{2} + 1} = \sqrt{4 - 2\sqrt{2}}$$
$$\operatorname{cosec} 22\tfrac{1}{2}° = \sqrt{1 + \cot^2 22\tfrac{1}{2}°} \quad = \sqrt{4 + 2\sqrt{2}}$$

Since $67\frac{1}{2}°$ is the complement of $22\frac{1}{2}°$, we have thus also derived the functions of $67\frac{1}{2}°$.

76. *To find the values of the sines and cosines of* $18°$ *and* $72°$.

If $x = 18°$, $2x$ is the complement of $3x$, and we shall have

$$\sin 2x = \cos 3x$$

Whence, $\quad 2\sin x\cos x = 4\cos^3 x - 3\cos x \qquad$ (Art. 71)

Dividing by $\cos x$, $\quad 2\sin x = 4\cos^2 x - 3$

$$= 4 - 4\sin^2 x - 3 \qquad \text{(Art. 19)}$$

We have then to solve the quadratic equation

$$4\sin^2 x + 2\sin x = 1$$

Thus, $$\sin x = \frac{-1 \pm \sqrt{1+4}}{4} = \frac{-1 \pm \sqrt{5}}{4}$$

Since 18° is in the first quadrant, its sine is positive; hence, the upper sign must be given to the radical, and we have

$$\sin 18° = \frac{\sqrt{5}-1}{4}$$

Therefore,

$$\cos 18° = \sqrt{1-\sin^2 18°} = \sqrt{\left(1 - \frac{6-2\sqrt{5}}{16}\right)}$$

$$= \sqrt{\left(\frac{10+2\sqrt{5}}{16}\right)} = \frac{\sqrt{10+2\sqrt{5}}}{4}$$

And, since 18° is the complement of 72°, we have

$$\sin 72° = \frac{\sqrt{10+2\sqrt{5}}}{4} \text{ and } \cos 72° = \frac{\sqrt{5}-1}{4}.$$

77. *To find the values of the sines and cosines of* 36° *and* 54°.

$$\cos 36° = \cos(2 \times 18°) = 1 - 2\sin^2 18° \qquad \text{(Art. 67)}$$

$$= 1 - 2\left(\frac{6-2\sqrt{5}}{16}\right) = \frac{4+4\sqrt{5}}{16} = \frac{\sqrt{5}+1}{4}$$

Therefore,

$$\sin 36° = \sqrt{1-\cos^2 36°} = \sqrt{\left(1 - \frac{6+2\sqrt{5}}{16}\right)}$$

$$= \sqrt{\left(\frac{10-2\sqrt{5}}{16}\right)} = \frac{\sqrt{10-2\sqrt{5}}}{4}$$

And, since 36° is the complement of 54°, we have

$$\sin 54° = \frac{\sqrt{5}+1}{4} \text{ and } \cos 54° = \frac{\sqrt{10-2\sqrt{5}}}{4}.$$

78. If A, B, and C denote the angles of a plane triangle, that is, if $A + B + C = 180°$, to prove the relations,

$$\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$$

$$\cos 2A + \cos 2B + \cos 2C = -1 - 4\cos A\cos B\cos C$$

Since $A + B + C = 180°$, we have

$$\left.\begin{aligned} B + C &= 180° - A \\ \frac{B+C}{2} &= 90° - \frac{A}{2} \end{aligned}\right\} \qquad (\mathbf{A})$$

Now, by (**27**) and (**49**),

$$\sin A + \sin B + \sin C$$

$$= 2\sin\frac{A}{2}\cos\frac{A}{2} + 2\sin\frac{B+C}{2}\cos\frac{B-C}{2}$$

which, by (**A**) and Art. 18,

$$= 2\cos\frac{B+C}{2}\cos\frac{A}{2} + 2\cos\frac{A}{2}\cos\frac{B-C}{2},$$

$$= 2\cos\frac{A}{2}\left(\cos\frac{B+C}{2} + \cos\frac{B-C}{2}\right)$$

$$= 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}, \qquad \text{by } (\mathbf{13}) \text{ and } (\mathbf{15})$$

Again, by (**31**) and (**51**),

$$\cos 2A + \cos 2B + \cos 2C$$

$$= -1 + 2\cos^2 A + 2\cos(B + C)\cos(B - C)$$

But, by (**A**), $\cos(B+C) = -\cos A$ (Art. 48)

Hence, we have

$$\cos 2A + \cos 2B + \cos 2C$$

$$= -1 - 2\cos A\cos(B+C) - 2\cos A\cos(B-C)$$

$$= -1 - 2\cos A\left[\cos(B + C) + \cos(B - C)\right]$$

$$= -1 - 4\cos A\cos B\cos C, \qquad \text{by } (\mathbf{13}) \text{ and } (\mathbf{15})$$

EXERCISES.

79. Prove the following relations:

1. $\sin(45°+y)=\dfrac{\sin y+\cos y}{\sqrt{2}}.$

2. $\tan(60°-y)=\dfrac{\sqrt{3}-\tan y}{1+\sqrt{3}\tan y}.$

3. $\operatorname{cosec} 2x=\tfrac{1}{2}\sec x \operatorname{cosec} x.$

4. $\dfrac{\sin 3A}{\sin A}-\dfrac{\cos 3A}{\cos A}=2.$

5. $\sin 4A=4\sin A\cos A-8\sin^3 A\cos A.$

6. $\cos 4A=1-8\cos^2 A+8\cos^4 A.$

7. $\sin x(1+\tan x)+\cos x(1+\cot x)=\sec x+\operatorname{cosec} x.$

8. $\tan x+\cot x=\dfrac{2}{\sin 2x}.$

9. $\dfrac{(1+\tan x)^2-(1-\tan x)^2}{(1+\tan x)^2+(1-\tan x)^2}=\sin 2x.$

10. $\sec 2x=1+\tan x\tan 2x.$

11. $\cot x-\tan x=2\cot 2x.$

12. $\sec^2 x\operatorname{cosec}^2 x=\sec^2 x+\operatorname{cosec}^2 x.$

13. $1+\cos 2x\cos 2y=2(\sin^2 x\sin^2 y+\cos^2 x\cos^2 y).$

14. $\cos y+\cos(120°+y)+\cos(120°-y)=0.$

15. $\tan(30°+y)\tan(30°-y)=\dfrac{2\cos 2y-1}{2\cos 2y+1}.$

16. $\cos^2(x+y)-\sin^2 x=\cos y\cos(2x+y).$

17. $\sin A\sin(B-C)+\sin B\sin(C-A)$
$+\sin C\sin(A-B)=0.$

18. $\sin(2x+y)-2\sin x\cos(x+y)=\sin y.$

19. $4\sin x\sin(60°+x)\sin(60°-x)=\sin 3x.$

20. $\sin 5x = 5 \sin x - 20 \sin^3 x + 16 \sin^5 x.$

21. $4 \sin^3 x + 4 \sin^3 (120° + x) + 4 \sin^3 (240° + x) = -3 \sin 3x.$

22. $\cos (A + B) \cos (A - B) + \cos (B + C) \cos (B - C)$
$+ \cos (C + A) \cos (C - A) = \cos 2A + \cos 2B + \cos 2C.$

23. $(\sin x + \sin y)^2 + (\cos x + \cos y)^2 = 4 \cos^2 \frac{x - y}{2}.$

24. $\frac{\cos x - \cos 3x}{\sin 3x - \sin x} = \tan 2x.$

25. $\cos^2 (x - y) - 2 \cos (x - y) \cos x \cos y = \sin^2 x + \sin^2 y - 1.$

If A, B, and C denote the angles of a plane triangle, prove the following relations:

26. $\cos A + \cos B + \cos C = 1 + 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}.$

27. $\tan A + \tan B + \tan C = \tan A \tan B \tan C.$

28. $\sin A + \sin B - \sin C = 4 \sin \frac{A}{2} \sin \frac{B}{2} \cos \frac{C}{2}.$

29. $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C.$

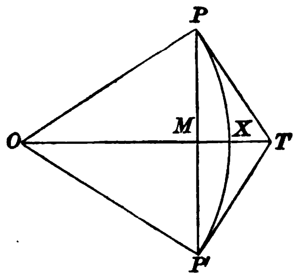
30. $\cos \frac{A}{2} + \cos \frac{B}{2} - \cos \frac{C}{2}$

$$= 4 \cos \left(45° + \frac{A}{4}\right) \cos \left(45° + \frac{B}{4}\right) \cos \left(45° - \frac{C}{4}\right).$$

V. MISCELLANEOUS THEOREMS.

30. *To find the limiting values of the fractions* $\frac{\sin x}{x}$ *and* $\frac{\tan x}{x}$ *when* x *is indefinitely decreased.*

Note. In such fractions as $\frac{\sin x}{x}$ and $\frac{\tan x}{x}$, x is supposed to be expressed in circular measure (Art. 5).



Let XOP and $P'OX$ be each equal to x, x being taken as an acute angle. With O as a centre, and any radius OX, describe the arc PXP'; draw TP and TP' tangents to this arc at P and P', respectively, and join PP'.

Then the right triangles OPT and $OP'T$ are equal, and we have $PT = P'T$. Also, $PM = P'M$, and OX is perpendicular to PP'.

By Geometry,

$$\text{Arc}\,PP' > \text{chord}\,PP', \text{ and } < PTP'$$

Hence, $\text{Arc}\,PX > PM \text{ and } < PT$

Therefore, $\dfrac{\text{Arc}\,PX}{OP} > \dfrac{PM}{OP} \text{ and } < \dfrac{PT}{OP}$

Or (Art. 5),

$$\text{circular measure of } x > \sin x \text{ and } < \tan x \qquad (\text{A})$$

Representing the circular measure of x by x simply, and dividing through by $\sin x$, we have

$$\frac{x}{\sin x} > 1 \text{ and } < \frac{\tan x}{\sin x} \text{ or } \frac{1}{\cos x}$$

That is, $\dfrac{\sin x}{x} < 1 \text{ and } > \cos x.$

But, as x decreases, $\cos x$ approaches 1 (Art. 36). Hence, at the limit, we have

$$\frac{\sin x}{x} = 1.$$

Again, $$\frac{\tan x}{x} = \frac{\sin x}{x \cos x} = \frac{\sin x}{x} \times \frac{1}{\cos x}$$

As x decreases indefinitely, $\frac{\sin x}{x}$ approaches the limit 1, as just proved, and $\frac{1}{\cos x}$ approaches 1. Hence, at the limit, we have

$$\frac{\tan x}{x} = 1.$$

It is evident from these results that the sines and tangents of very small angles are approximately proportional to the angles themselves.

81. *To find the limiting value of* $\left(\cos\frac{x}{n}\right)^n$, *when* n *is indefinitely increased.*

$$\left(\cos\frac{x}{n}\right)^n = \left(\cos^2\frac{x}{n}\right)^{\frac{n}{2}} = \left(1 - \sin^2\frac{x}{n}\right)^{\frac{n}{2}}$$

Expanding by the Binomial Theorem, we obtain

$$\left(\cos\frac{x}{n}\right)^n = 1 - \frac{n}{2}\sin^2\frac{x}{n} + \frac{\frac{n}{2}\left(\frac{n}{2}-1\right)}{\underline{|2}}\sin^4\frac{x}{n} - \cdots$$

$$= 1 - \frac{n}{2}\sin^2\frac{x}{n} + \frac{n^2}{4}\left(\frac{1}{2} - \frac{1}{n}\right)\sin^4\frac{x}{n} - \cdots$$

$$= 1 - \frac{x^2}{2n} \cdot \frac{\sin^2\frac{x}{n}}{\frac{x^2}{n^2}} + \left(\frac{1}{2} - \frac{1}{n}\right)\frac{x^4}{4n^2} \cdot \frac{\sin^4\frac{x}{n}}{\frac{x^4}{n^4}} - \cdots$$

$$= 1 - \frac{x^2}{2n}\left(\frac{\sin\frac{x}{n}}{\frac{x}{n}}\right)^2 + \left(\frac{1}{2} - \frac{1}{n}\right)\frac{x^4}{4n^2}\left(\frac{\sin\frac{x}{n}}{\frac{x}{n}}\right)^4 - \cdots$$

When n is indefinitely increased, $\frac{x}{n}$ is indefinitely decreased, and consequently $\frac{\sin\frac{x}{n}}{\frac{x}{n}}$ approaches the limit 1 (Art. 80).

Also, $\frac{x^2}{2n}$, $\frac{x^4}{4n^2}$, $\cdots$, approach the limit 0, and $\frac{1}{2}-\frac{1}{n}$ and all similar terms approach finite limits. Hence, at the limit, every term disappears except the first, and we have

$$\left(\cos\frac{x}{n}\right)^n = 1.$$

INVERSE TRIGONOMETRICAL FUNCTIONS.

82. The expression $\sin^{-1}y$ is called the *inverse sine* of y, or the *anti-sine* of y, and it denotes an angle whose sine is equal to y.

Thus, the fact that the sine of an angle x is equal to y may be expressed in either of the ways, $\sin x = y$ or $x = \sin^{-1}y$.

Similarly, $\cos^{-1}y$ denotes an angle whose cosine is equal to y; $\tan^{-1}y$, an angle whose tangent is equal to y, etc.

Note. The student must be careful not to confuse this notation with the *exponent* -1. The -1 power of $\sin x$ is expressed $(\sin x)^{-1}$ and not $\sin^{-1}x$.

1. Prove the relation $3\sin^{-1}a = \sin^{-1}(3a - 4a^3)$.

By (46), Art. 71, $\sin 3x = 3\sin x - 4\sin^3 x$ (A)

Let $\sin x = a$. Then, by Art. 82,

$$x = \sin^{-1}a \text{ and } 3x = 3\sin^{-1}a$$

Substituting these values in (A), we have

$$\sin(3\sin^{-1}a) = 3a - 4a^3$$

Whence,

$$3\sin^{-1}a = \sin^{-1}(3a - 4a^3)$$

2. Prove that one value of $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3}$ is $\frac{\pi}{4}$.

Let

$$\left.\begin{aligned}\tan^{-1}\frac{1}{2} &= x\\ \tan^{-1}\frac{1}{3} &= y\end{aligned}\right\}\text{ whence, }\left\{\begin{aligned}\tan x &= \frac{1}{2}\\ \tan y &= \frac{1}{3}\end{aligned}\right.$$

But,

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

Substituting the values of $\tan x$ and $\tan y$, we have

$$\tan(x+y)=\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{6}}=\frac{\frac{5}{6}}{\frac{5}{6}}=1$$

Hence, one value of $x+y$ is $\frac{\pi}{4}$ (Art. 17); that is, one value of $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}$ is $\frac{\pi}{4}$.

3. Prove that one value of $\cos^{-1}\sqrt{\frac{2}{3}}+\cos^{-1}\frac{3+\sqrt{6}}{6}$ is $\frac{\pi}{3}$.

$$\text{Let}\quad \left.\begin{aligned}\cos^{-1}\sqrt{\frac{2}{3}}&=x\\ \cos^{-1}\frac{3+\sqrt{6}}{6}&=y\end{aligned}\right\}\text{ whence, }\left\{\begin{aligned}\cos x&=\sqrt{\frac{2}{3}}\\ \cos y&=\frac{3+\sqrt{6}}{6}\end{aligned}\right.$$

$$\text{Hence,}\quad \sin x=\sqrt{1-\cos^2 x}=\sqrt{1-\frac{2}{3}}=\frac{1}{\sqrt{3}}$$

$$\text{and}\quad \sin y=\sqrt{\left(1-\frac{15+6\sqrt{6}}{36}\right)}=\frac{\sqrt{21-6\sqrt{6}}}{6}$$

$$=\frac{3\sqrt{2}-\sqrt{3}}{6}$$

$$\text{Hence,}\quad \cos(x+y)=\sqrt{\frac{2}{3}}\cdot\frac{3+\sqrt{6}}{6}-\frac{1}{\sqrt{3}}\cdot\frac{3\sqrt{2}-\sqrt{3}}{6}$$

$$=\frac{3\sqrt{2}+2\sqrt{3}-3\sqrt{2}+\sqrt{3}}{6\sqrt{3}}=\frac{3\sqrt{3}}{6\sqrt{3}}=\frac{1}{2}$$

Therefore, one value of $x+y$ is 60° or $\frac{\pi}{3}$; that is, one value of $\cos^{-1}\sqrt{\frac{2}{3}}+\cos^{-1}\frac{3+\sqrt{6}}{6}$ is $\frac{\pi}{3}$.

EXERCISES.

4. From the formula $\sin 2x=2\sin x\cos x$ prove the relation

$$2\sin^{-1}a=\sin^{-1}(2a\sqrt{1-a^2}).$$

5. From the formula $\sin(x+y)=\sin x\cos y+\cos x\sin y$ prove the relation

$$\sin^{-1}a+\sin^{-1}b=\sin^{-1}(a\sqrt{1-b^2}+b\sqrt{1-a^2}).$$

6. From the formula $\tan 2x = \frac{2\tan x}{1-\tan^2 x}$ prove the relation

$$2\tan^{-1} a = \tan^{-1}\frac{2a}{1-a^2}.$$

7. From the formula $\tan x + \cot x = 2\operatorname{cosec} 2x$ (see Ex. 8, Art. 79), prove the relation

$$2\tan^{-1} a = \operatorname{cosec}^{-1}\left(\frac{a^2+1}{2a}\right).$$

8. Prove that one value of $\tan^{-1}\frac{1}{7} + 2\tan^{-1}\frac{1}{3}$ is $\frac{\pi}{4}$.

9. Prove that one value of $\cot^{-1}\frac{3}{4} + \cot^{-1}\frac{1}{7}$ is $\frac{3\pi}{4}$.

10. Prove that one value of $\sin^{-1}\frac{\sqrt{5}}{5} + \cot^{-1} 3$ is $\frac{\pi}{4}$.

11. Prove that one value of $\cos^{-1}\frac{9}{\sqrt{82}} + \operatorname{cosec}^{-1}\frac{\sqrt{41}}{4}$ is $\frac{\pi}{4}$.

12. Prove that $\tan(2\tan^{-1} a) = 2\tan(\tan^{-1} a + \tan^{-1} a^3)$.

SOLUTION OF TRIGONOMETRIC EQUATIONS.

83. 1. Solve the equation $2\sin^2 x - 5\cos x - 4 = 0$.

By Art. 19, $2 - 2\cos^2 x - 5\cos x - 4 = 0$

That is, $2\cos^2 x + 5\cos x = -2$

Whence,
$$\cos x = \frac{-5 \pm \sqrt{25-16}}{4}$$
$$= \frac{-5 \pm 3}{4} = -2 \text{ or } -\frac{1}{2}$$

The result -2 is inapplicable, as, by Art. 39, the value of the cosine of an angle must lie between 1 and -1. Thus, we have

$$\cos x = -\frac{1}{2}.$$

One value of x is 120° or $\frac{2\pi}{3}$ (Art. 51). Hence, by Art. 52,

$$x = 2n\pi \pm \frac{2\pi}{3}.\quad \textit{Ans.}$$

2. Solve the equation $\tan 2x = 6 \tan x$.

By Art. 67, $$\frac{\tan x}{1-\tan^2 x} = 3 \tan x.$$

One solution is evidently $\tan x = 0$; in this the general value of x is $n\pi$ (Art. 52). Dividing by $\tan x$, we have

$$\frac{1}{1-\tan^2 x} = 3$$

That is, $$1 = 3 - 3\tan^2 x$$

Or, $$\tan^2 x = \frac{2}{3}$$

Whence, $$\tan x = \pm \tfrac{1}{3}\sqrt{6}$$

Hence, $$x = \tan^{-1}(\pm \tfrac{1}{3}\sqrt{6})$$

Ans. $x = n\pi$ or $\pm \tan^{-1}(\frac{1}{3}\sqrt{6})$.

EXAMPLES.

In the following, find the values of x:

3. $\sin x = \sin 2x$.

4. $\sin x = \sin 3x$.

5. $\cos x \cot x = 1$.

6. $\sin 2x = \cot x$.

7. $\cos 2x = 2 \sin x$.

8. $\tan(45° - x) + \cot(45° - x) = 4$.

9. $\sin(a + x) = \cos(a - x)$.
(Divide by $\cos x$.)

10. $\sin x = \sin a \sin(x + b)$.

VI. LOGARITHMS.

84. *The logarithm of a quantity to a given base is the exponent of the power to which the base must be raised to equal the quantity.*

Thus, if $a^x = m$, x is the exponent of the power to which the base a must be raised to equal the quantity m; that is, x is the logarithm of m to the base a; which is briefly expressed

$$x = \log_a m$$

85. To illustrate, suppose $a = 3$.

Then, since

$$\begin{array}{ll}
3^0 = 1, & 0 = \log_3 1 \\
3^1 = 3, & 1 = \log_3 3 \\
3^2 = 9, & 2 = \log_3 9 \\
3^{-1} = \dfrac{1}{3}, & -1 = \log_3 \dfrac{1}{3} \\
3^{-2} = \dfrac{1}{3^2} = \dfrac{1}{9}, & -2 = \log_3 \dfrac{1}{9}
\end{array}$$

Hence, in the system whose base is 3, $\log 1 = 0$, $\log 3 = 1$, $\log 9 = 2$, $\log \frac{1}{3} = -1$, $\log \frac{1}{9} = -2$, etc.

Again, suppose $a = 10$.

Then, since

$$\begin{array}{ll}
10^0 = 1, & 0 = \log_{10} 1 \\
10^1 = 10, & 1 = \log_{10} 10 \\
10^2 = 100, & 2 = \log_{10} 100 \\
10^3 = 1000, & 3 = \log_{10} 1000 \\
10^{-1} = \frac{1}{10} = .1, & -1 = \log_{10} .1 \\
10^{-2} = \frac{1}{100} = .01, & -2 = \log_{10} .01 \\
10^{-3} = \frac{1}{1000} = .001, & -3 = \log_{10} .001
\end{array}$$

That is, in the system whose base is 10, $\log 1 = 0$, $\log 10 = 1$, $\log 100 = 2$, $\log 1000 = 3$, $\log .1 = -1$, $\log .01 = -2$, $\log .001 = -3$, etc.

86. The only system in practical use for numerical computations is the Common System or Briggs' System, in which the base is 10. Hence, the definition of the common logarithm of a quantity is *the exponent of that power of* 10 *which equals the quantity.*

87. It is customary in common logarithms to omit the subscript 10, which denotes the base; hence we may write the result of Art. 85 as follows:

$$\begin{array}{rr}
\log 1 = 0 & \log .1 = -1 = 9 - 10 \\
\log 10 = 1 & \log .01 = -2 = 8 - 10 \\
\log 100 = 2 & \log .001 = -3 = 7 - 10 \\
\log 1000 = 3 & \text{etc.}
\end{array}$$

The second form of the results in the second column is preferable in the solution of examples.

88. If a number is not an exact power of 10, its common logarithm can only be expressed approximately; the integral part of the logarithm is called the *characteristic*, and the decimal part the *mantissa*.

Thus, we find from the table,

$$\log 13 = 1.113943.$$

This signifies that approximately $10^{1.113943} = 13$. In this case the characteristic is 1, and the mantissa is .113943.

89. It is evident from the first column of Art. 87 that the logarithm of any number between 1 and 10 is equal to 0 plus some decimal; between 10 and 100 is equal to 1 plus some decimal; between 100 and 1000 is equal to 2 plus some decimal; etc.

That is, the characteristic of the logarithm of any number with *one* figure to the left of its decimal point is 0; with *two* figures to the left of the point is 1; with *three* figures to the left of the point is 2; etc.

90. Similarly, from the second column of Art. 87 we see that the logarithm of any number between 1 and .1 lies between 0 and $9 - 10$, or is equal to 9 plus some decimal -10; of any number between .1 and .01 is equal to 8 plus some decimal -10; between .01 and .001 is equal to 7 plus some decimal -10; etc.

That is, the characteristic of the logarithm of any decimal with *no* zeros between the decimal point and first figure is

9, with -10 after the mantissa; of any decimal with *one* zero between its point and first figure is 8, with -10 after the mantissa; with *two* zeros between its point and first figure is 7, with -10 after the mantissa; etc.

91. For reasons which will be given hereafter only the mantissa of the logarithm is given in the tables; the characteristic must be supplied by the reader. The rules for characteristic are based on the last portions of Arts. 89 and 90:

I. *If the number is greater than unity, the characteristic is* 1 *less than the number of figures to the left of the decimal point.*

II. *If the number is less than unity, the characteristic is found by subtracting the number of zeros between the decimal point and first significant figure from* 9; *writing* -10 *after the mantissa.*

For example, characteristic of log 906328.5 $= 5$;

characteristic of log .00792 $= 7$, with -10 after the mantissa.

It is customary in practice to omit the -10 after the mantissa; it should be remembered, however, that it is really a part of the logarithm, and should be allowed for, and subjected to precisely the same operations as the rest of the logarithm. Beginners will find it useful to write it in all cases; and in some problems it cannot conveniently be omitted.

Note. Many writers, in dealing with the characteristics of the logarithms of numbers less than unity, combine the two portions of the characteristic, writing the result as a *negative characteristic* before the mantissa. Thus, instead of such an expression as $7.603582-10$, the student will frequently find $\bar{3}.603582$; a minus sign being written *over* the characteristic to denote that it alone is negative, the mantissa being always positive. The objection to this notation is the inconvenience of using numbers partly positive and partly negative.

PROPERTIES OF LOGARITHMS.

92. *In any system the logarithm of unity is zero.*

For, since $a^0 = 1$, for any value of a, $0 = \log_a 1$.

93. *In any system the logarithm of the base itself is unity.*

For, since $a^1 = a$, for any value of a, $1 = \log_a a$.

94. *In any system whose base is greater than unity, the logarithm of zero is minus infinity.*

For since, when $a > 1$, $a^{-\infty} = \frac{1}{a^{\infty}} = \frac{1}{\infty} = 0$, $-\infty = \log_a 0$.

If the base is less than unity, the logarithm of 0 is $+\infty$.

For since, when $a < 1$, $a^{\infty} = 0$, $\infty = \log_a 0$.

95. *In any system the logarithm of the product of any number of factors is equal to the sum of the logarithms of those factors.*

Assume the equations

$$\left.\begin{array}{l} a^x = m \\ a^y = n \\ a^z = p \\ \text{etc.} \end{array}\right\} \text{whence, by Art. 84,} \left\{\begin{array}{l} x = \log_a m \\ y = \log_a n \\ z = \log_a p \\ \text{etc.} \end{array}\right.$$

Multiplying, $a^x \times a^y \times a^z \times \cdots = mnp \cdots$

That is, $a^{x+y+z+\cdots} = mnp \cdots$

Whence, $x + y + z + \cdots = \log_a mnp \cdots$

Substituting the values of x, y, z, etc.,

$$\log_a mnp \cdots = \log_a m + \log_a n + \log_a p + \cdots$$

By the application of this theorem we may find the logarithm of any composite number, provided we know the logarithm of each of its factors.

For example, given $\log 2 = .301030$, $\log 3 = .477121$, required $\log 72$.

$$\begin{aligned}\log 72 &= \log(2 \times 2 \times 2 \times 3 \times 3)\\ &= \log 2 + \log 2 + \log 2 + \log 3 + \log 3\\ &= 3 \times \log 2 + 2 \times \log 3\\ &= .903090 + .954242 = 1.857332. \textit{ Ans.}\end{aligned}$$

EXAMPLES.

Given $\log 2 = 0.301030$, $\log 3 = 0.477121$, $\log 7 = 0.845098$, calculate:

1. $\log 48$.
2. $\log 441$.
3. $\log 56$.
4. $\log 98$.
5. $\log 84$.
6. $\log 567$.
7. $\log 168$.
8. $\log 7056$.
9. $\log 504$.
10. $\log 3087$.
11. $\log 15552$.
12. $\log 14406$.

96. *In any system the logarithm of a fraction is equal to the logarithm of the numerator minus the logarithm of the denominator.*

Assume the equations

$$\left.\begin{aligned} a^x &= m\\ a^y &= n\end{aligned}\right\} \text{ whence, } \left\{\begin{aligned} x &= \log_a m\\ y &= \log_a n\end{aligned}\right.$$

Dividing, $$\frac{a^x}{a^y} = \frac{m}{n}, \text{ or } a^{x-y} = \frac{m}{n}$$

Whence, $$x - y = \log_a \frac{m}{n}$$

Substituting the values of x and y,

$$\log_a \frac{m}{n} = \log_a m - \log_a n$$

By this theorem, a logarithm being given, we may derive certain others from it. For instance, if we know $\log 2 = 0.301030$, then

$$\log 5 = \log \frac{10}{2} = \log 10 - \log 2 = 1. - 0.301030 = 0.698970.$$

EXAMPLES.

Given $\log 2 = 0.301030$, $\log 3 = 0.477121$, $\log 7 = 0.845098$, calculate:

1. $\log 15$.
2. $\log 125$.
3. $\log \frac{10}{7}$.
4. $\log 175$.
5. $\log 3\frac{1}{8}$.
6. $\log 11\frac{1}{9}$.
7. $\log 7\frac{1}{7}$.
8. $\log \frac{35}{3}$.
9. $\log 5\frac{4}{9}$.

97. *In any system the logarithm of any power of a quantity is equal to the logarithm of the quantity multiplied by the exponent of the power.*

Assume the equation

$$a^x = m, \quad \text{whence,} \quad x = \log_a m$$

Raising both members of the assumed equation to the pth power,

$$a^{px} = m^p$$

whence,

$$px = \log_a m^p$$

Substituting the value of x,

$$\log_a m^p = p \log_a m$$

Example. Given $\log 2 = .301030$, required the logarithm of $2^{\frac{5}{3}}$.

$$\log 2^{\frac{5}{3}} = \tfrac{5}{3} \log 2 = \frac{5}{3} \times .301030 = .501717. \quad \textit{Ans.}$$

EXAMPLES.

Given $\log 2 = .301030$, $\log 3 = .477121$, calculate:

1. $\log 3^{\frac{2}{5}}$.
2. $\log 2^{12}$.
3. $\log 12^{-5}$.
4. $\log 3^{-\frac{2}{9}}$.
5. $\log 432^{\frac{1}{5}}$.
6. $\log 1296^{-\frac{4}{5}}$.

98. *In any system the logarithm of any root of a quantity is equal to the logarithm of the quantity divided by the index of the root.*

For,
$$\log_a \sqrt[r]{m} = \log_a (m^{\frac{1}{r}}) = \frac{1}{r} \log_a m \qquad \text{(Art. 97)}$$

Example. Given $\log 2 = .301030$, required the logarithm of $\sqrt[5]{2}$.

$$\log \sqrt[5]{2} = \tfrac{1}{5} \log 2 = \frac{1}{5} \times .301030 = .060206. \quad \textit{Ans.}$$

EXAMPLES.

Given $\log 2 = .301030$, $\log 3 = .477121$, calculate:

1. $\log \sqrt[7]{3}$.
2. $\log \sqrt[3]{24}$.
3. $\log \sqrt[8]{648}$.
4. $\log \sqrt{75}$.
5. $\log \sqrt[3]{1125}$.
6. $\log \sqrt[4]{16875}$.

99. *In the common system, the mantissæ of the logarithms of all numbers having the same sequence of figures will be the same.*

For example, suppose that $\log 3.053 = .484727$.

Then,
$$\begin{aligned} \log 30.53 &= \log (3.053 \times 10) = \log 3.053 + \log 10 \\ &= .484727 + 1 = 1.484727. \\ \log 30530 &= \log (3.053 \times 10000) = \log 3.053 + \log 10000 \\ &= .484727 + 4 = 4.484727. \\ \log .03053 &= \log (3.053 \times .01) = \log 3.053 + \log .01 \\ &= .484727 + 8 - 10 = 8.484727 - 10; \text{ etc.} \end{aligned}$$

It is clear, then, that if a number be multiplied or divided by any integral power of 10, thereby producing another number having the same sequence of figures, the mantissæ of their logarithms will be the same.

Or, to illustrate, if $\log 3.053 = .484727$, then

$$\begin{aligned} \log 30.53 &= 1.484727 & \log .3053 &= 9.484727 - 10 \\ \log 305.3 &= 2.484727 & \log .03053 &= 8.484727 - 10 \\ \log 3053 &= 3.484727 & \log .003053 &= 7.484727 - 10 \\ &\text{etc.} & &\text{etc.} \end{aligned}$$

We may now see the reason why, as stated in Art. 91, only the mantissæ are given in the table; for, if we wish to find the logarithm of any number, we have only to find the mantissa of the sequence of figures composing it from the table, and we can then prefix the proper characteristic, depending on the position of the decimal point, in accordance with the rules of Art. 91.

This property of logarithms is only enjoyed by the common system, and constitutes its superiority over all others for numerical computations.

100. *Given the logarithm of a quantity to a certain base, to calculate its logarithm to any other base.*

Assume the equations

$$\left.\begin{array}{l} a^x = m \\ b^y = m \end{array}\right\} \text{ whence, } \left\{\begin{array}{l} x = \log_a m \\ y = \log_b m \end{array}\right.$$

From the assumed equations,

$$a^x = b^y, \quad \text{or} \quad a^{\frac{x}{y}} = b$$

Whence,

$$\frac{x}{y} = \log_a b, \text{ or } y = \frac{x}{\log_a b}$$

Substituting the values of x and y,

$$\log_b m = \frac{\log_a m}{\log_a b}$$

That is, if we know the logarithm of m to a certain base, a, its logarithm to any other base, b, is found by dividing by the logarithm of b to the base a.

101. *To show that* $\log_a b \times \log_b a = 1$, *for any values of* a *and* b.

Putting $m = a$ in the result of Art. 100,

$$\log_b a = \frac{\log_a a}{\log_a b} = \frac{1}{\log_a b} \qquad \text{(Art. 93)}$$

Whence,

$$\log_a b \times \log_b a = 1.$$

The theorem may be proved independently as follows:

Assume the equation,

$$a^x = b, \quad \text{whence} \quad x = \log_a b$$

Then, $$b^{\frac{1}{x}} = a, \quad \text{whence} \quad \frac{1}{x} = \log_b a$$

Therefore, $$\log_a b \times \log_b a = x \times \frac{1}{x} = 1$$

102. The following examples illustrate the applications of Arts. 95, 96, 97, and 98.

1. $\log\left(\frac{a}{b}\right)^{\frac{c}{d}} = \frac{c}{d}\log\frac{a}{b}$ (Art. 97)

$= \frac{c}{d}(\log a - \log b)$ (Art. 96)

2. $\log\frac{\sqrt[n]{a} \times \sqrt[m]{b}}{\sqrt[p]{c}} = \log(\sqrt[n]{a} \times \sqrt[m]{b}) - \log\sqrt[p]{c}$ (Art. 96)

$= \log\sqrt[n]{a} + \log\sqrt[m]{b} - \log\sqrt[p]{c}$ (Art. 95)

$= \frac{1}{n}\log a + \frac{1}{m}\log b - \frac{1}{p}\log c$ (Art. 98)

Reduce the following to their simplest forms:

3. $\log\frac{abc}{de}$.

4. $\log(\sqrt[n]{a} \times b^3 \times c^{\frac{d}{2}})$.

5. $\log\frac{2^{\frac{2}{3}}}{3^{\frac{5}{6}}}$.

6. $\log\sqrt[n]{\frac{a^2}{bc}}$.

7. $\log\frac{\sqrt[n]{ab}}{\sqrt[m]{c}}$.

8. $\log\frac{\sqrt[4]{a}}{bc^{\frac{2}{3}}d^2}$.

9. $\log\left(\sqrt[5]{\frac{a}{b}} \div (cd)^{-\frac{m}{n}}\right)$.

USE OF THE TABLE OF LOGARITHMS OF NUMBERS.

103. The first table (Appendix, pages 1 to 16) gives the mantissæ of the logarithms of all numbers from 1 to 10000, calculated to six decimal places. On the first page of the table are the logarithms of the numbers between 1 and 100. This table is added simply for convenience, as the same mantissæ are to be found in the other part of the table.

To find the logarithm of any number consisting of four figures.

Find, in the column headed N, the first three figures of the given number. Then the mantissa of the required logarithm will be found in the horizontal line corresponding, in the vertical column which has the fourth figure of the given number at the top. If only the last four figures of the mantissa are found, the first two figures may be obtained from the nearest mantissa above, in the same vertical column, which consists of six figures. Finally, prefix the proper characteristic (Art. 91).

For example,
$$\log 140.8 = 2.148603$$
$$\log .05837 = 8.766190 - 10$$
$$\log 8516 = 3.930236$$

For a number consisting of one or two figures, use the first page of the table, which needs no explanation; for a number of three figures, look in the column headed N, and take the mantissa corresponding in the column headed 0. For example, $\log 94.6 = 1.975891$.

104. *To find the logarithm of a number of more than four figures.*

For example, let it be required to find $\log 3296.78$.

From the table, we find
$$\log 3296 = 3.517987$$
$$\log 3297 = 3.518119$$

That is, an increase of one unit in the number produces an increase of .000132 in the logarithm. Then, evidently, an increase of .78 unit in the number will produce an increase of $.78 \times .000132$ in the logarithm $= .000103$ to the nearest sixth decimal place.

Therefore,

$$\begin{aligned}\log 3296.78 &= \log 3296 + .000103\\ &= 3.517987 + .000103 = 3.518090. \quad \textit{Ans.}\end{aligned}$$

Note. The foregoing method is based upon the assumption that the differences of logarithms are proportional to the differences of their corresponding numbers, which is not strictly correct, but sufficiently exact for practical purposes.

We derive the following rule from the above operation:

Find in the table the mantissa of the first four significant figures, without regard to the position of the decimal point.

Find the difference between this and the mantissa of the next higher number of four figures; (called the *tabular difference*, and to be found in the column headed D on each page; see Note below).

Multiply the tabular difference by the rest of the figures of the given number, with a decimal point before them.

Add the result to the mantissa of the first four figures.

Prefix the proper characteristic.

1. Find the logarithm of .02243076.

$$\begin{array}{rr}
 & \text{Mantissa of } 2243 = 350829\\
\text{Tabular difference} = 194 & \underline{15}\\
\underline{.076} & 350844\\
1.164 & \\
\underline{13.58} & \\
\text{Correction} = 14.744 = 15, \text{ nearly.} &
\end{array}$$

Ans. 8.350844 − 10.

Note. To find the tabular difference mentally, subtract the last figure of the mantissa from the last figure of the next greater, and take the nearest whole number ending in that figure to the number in the column headed D in the same line. For instance, in finding log .02243076, the last figure of the mantissa of 2243 is 9, and of the next greater mantissa, 3; 9 from 13 leaves 4, and the nearest number ending in 4 to 193, the number in the column headed D, is 194, the proper tabular difference.

EXAMPLES.

Find the logarithms of the following numbers:

2. .053.
3. 51.8.
4. .2956.
5. 1.0274.
6. 33.6908.
7. .0602851.
8. 65000.63.
9. .001030741.
10. 912.255.
11. .876092.
12. 7303.078.
13. .0436927.

14. Given log 7.83 = .89376, log 7.84 = .89432; find log 78309.

15. Given log .05229 = 8.718419 − 10, log .05230 = 8.718502 − 10; find log 52.2938.

16. Given log 315.08 = 2.4984208, log 315.09 = 2.4984346; find log .003150823.

17. Given log 18.84 = 1.275081, log 18.87 = 1.275772; find log .188527.

18. Given log 9.5338 = .9792660, log 9.5342 = .9792843; find log 95.34071.

105. *To find the number corresponding to a logarithm.*

For example, let it be required to find the number whose logarithm is 3.693845.

Since the characteristic depends only on the position of the decimal point, and in no way affects the *sequence of figures* corresponding, we can obtain all of the number corresponding, except the decimal point, by considering the mantissa only. We find in the table the mantissa 693815, of which the corresponding number is 4941, and the mantissa 693903, of which the corresponding number is 4942.

That is, an increase of 88 in the mantissa produces an increase of one unit in the number corresponding. Hence, an increase of 30 in the mantissa will produce an increase of $\frac{30}{88}$ of a unit in the number, or .34, nearly. Therefore,

Number corresponding = 4941 + .34 = 4941.34. *Ans.*

We base the following rule on the above operation:

Find in the table the next less mantissa, the four figures corresponding, and the tabular difference.

Subtract the next less from the given mantissa.

Divide the remainder by the tabular difference; (the quotient in general cannot be depended upon to more than two decimal places).

Annex all of the quotient except the decimal point to the first four figures of the number.

Point off.

Note. The rules for pointing off are the reverse of the rules for characteristic given in Art. 91.

I. *If* −10 *is not written after the mantissa, add* 1 *to the characteristic, giving the number of figures to the left of the decimal point.*

II. *If* −10 *is written after the mantissa, subtract the characteristic from* 9; *giving the number of zeros to be placed between the decimal point and first figure.*

1. Find the number whose logarithm is 7.950185−10.

950185
Next less mantissa = 950170; four figures corresponding = 8916
Tabular difference = 49) 15.00 (.31, nearly.
147
30

Therefore, number corresponding = .00891631. *Ans.*

EXAMPLES.

Find the numbers corresponding to the following:

2. 1.880814.
3. 9.470410−10.
4. 0.820204.
5. 4.745126.
6. 8.044891−10.
7. 2.270293.
8. 9.350064−10.
9. 3.000027.
10. 0.990191.
11. 7.115658−10.
12. 8.535003−10.
13. 1.670180.

14. Given $\log 113 = 2.05308$, $\log 114 = 2.05690$; find number corresponding to 1.05411.

15. Given $\log .08630 = 8.936011 - 10$, $\log .08631 = 8.936061 - 10$; find number corresponding to 0.936049.

16. Given $\log 2.0702 = .3160123$, $\log 2.0703 = .3160333$; find number corresponding to $9.3160138 - 10$.

17. Given $\log 548.3 = 2.739018$, $\log 548.9 = 2.739493$; find number corresponding to $7.739416 - 10$.

18. Given $\log 7.3488 = .8662164$, $\log 7.3492 = .8662401$; find number corresponding to 2.8662350.

106. In the application of Arts. 95, 96, 97, and 98, we have to perform the operations of Addition, Subtraction, Multiplication, and Division with logarithms. As some of the problems which may arise are peculiar, we give a few hints as to their solution, which will be found of service.

1. ADDITION. If, in the sum, -10, -20, -30, etc., are written after the mantissa, and the characteristic standing before the mantissa is greater than 9, subtract from both parts of the logarithm such a multiple of 10 as will make the characteristic before the mantissa less than 10.

For example, $13.354802 - 10$ should be changed to 3.354802; $28.964316 - 30$ should be changed to $8.964316 - 10$; etc.

2. SUBTRACTION. In subtracting a greater logarithm from a less, or in subtracting a negative logarithm from a positive, the characteristic of the minuend should be increased by 10, -10 being written after the mantissa to compensate.

For example, to subtract 3.121468 from 2.503964, we write the minuend in the form $12.503964 - 10$; subtracting from this 3.121468, we have as a result $9.382496 - 10$.

To subtract $9.635321 - 10$ from $9.583427 - 10$, we write the minuend in the form $19.583427 - 20$; subtracting from this $9.635321 - 10$, we have as a result $9.948106 - 10$.

3. Multiplication. The hint already given for reducing the result of Addition applies with equal force to Multiplication.

To multiply a logarithm by a fraction, multiply first by the numerator, and divide the result by the denominator.

4. Division. In dividing a negative logarithm, add to both parts of the logarithm such a multiple of 10 as will make the quantity after the mantissa exactly divisible by the divisor, with -10 as the quotient.

For example, to divide $7.402938-10$ by 6, we add 50 to both parts of the logarithm, giving $57.402938-60$. Dividing this by 6, we have as a result $9.567156-10$.

EXAMPLES.

1. Add $9.096004-10$, 4.581726, and $8.447510-10$.
2. Add $7.196070-10$, $8.822209-10$, and 2.205683.
3. Subtract 0.659321 from 0.511490.
4. Subtract $7.901338-10$ from 1.009800.
5. Subtract $9.156243-10$ from $8.750404-10$.
6. Multiply $9.105107-10$ by 3.
7. Divide $8.452633-10$ by 4.
8. Divide $9.670392-10$ by 11.
9. Multiply $9.668311-10$ by $\frac{2}{7}$.

SOLUTIONS OF ARITHMETICAL PROBLEMS BY LOGARITHMS.

107. In finding the value of any arithmetical quantity by logarithms, we first find the logarithm of the quantity, as in Art. 102, by the aid of the table, and then find the number corresponding to the result.

1. Find the value of $.0631 \times 7.208 \times 512.72$.

By Art. 95, $\log(.0631 \times 7.208 \times 512.72)$

$$= \log .0631 + \log 7.208 + \log 512.72$$

$$\begin{aligned} \log .0631 &= 8.800029 - 10 \\ \log 7.208 &= 0.857815 \\ \log 512.72 &= 2.709880 \end{aligned}$$

Adding, $\therefore \log$ of Ans. $= 12.367724 - 10$

$= 2.367724$ (Art. 106, 1)

Number corresponding to $2.367724 = 233.197$. *Ans.*

2. Find the value of $\dfrac{3368.52}{7980.04}$.

$$\log \frac{3368.52}{7980.04} = \log 3368.52 - \log 7980.04$$

$\log 3368.52 = 13.527439 - 10$ (Art. 106, 2)

$\log 7980.04 = 3.902005$

Subtracting, $\therefore \log$ of Ans. $= 9.625434 - 10$

Number corresponding $= .422118$. *Ans.*

3. Find the value of $(.0980937)^5$.

$$\log(.0980937)^5 = 5 \times \log .0980937$$

$$\begin{aligned} \log .0980937 &= 8.991641 - 10 \\ &\ 5 \end{aligned}$$

Multiplying, $\therefore \log$ of Ans. $= 44.958205 - 50$

$= 4.958205 - 10$

Number corresponding $= .0000090825$. *Ans.*

4. Find the value of $\sqrt[7]{.001}$.

$$\log \sqrt[7]{.001} = \tfrac{1}{7} \log .001$$

$\log .001 = 7 - 10$ (Art. 87)

$= 67 - 70$ (Art. 106, 4)

Dividing by 7, $\therefore \log$ of Ans. $= 9.571429 - 10$

Number corresponding $= .37276$. *Ans.*

5. Find the value of $\frac{2\sqrt[3]{5}}{3^{\frac{5}{6}}}$.

$$\log\frac{2\sqrt[3]{5}}{3^{\frac{5}{6}}} = \log 2 + \tfrac{1}{3}\log 5 - \tfrac{5}{6}\log 3$$

$$\begin{array}{lr}
 & \log 2 = .301030 \\
\log 5 = .698970; & \text{divide by } 3 = \underline{.232990} \\
 & .534020 \\
\log 3 = .477121; & \text{multiply by } \tfrac{5}{6} = \underline{.397601} \\
\text{Subtracting,} & \therefore \log \text{ of Ans.} = .136419 \\
\text{Number corresponding} & = 1.36905.\ \textit{Ans.}
\end{array}$$

The work of the next two examples will be exhibited in the customary form, the -10's being omitted after the mantissæ. See Art. 91.

6. Find the value of $\sqrt[7]{.00003591}$.

$$\begin{array}{rl}
\log\sqrt[7]{.00003591} & = \tfrac{1}{7}\log .00003591 \\
\log .00003591 & = 5.555215 \\
 & 7\underline{)5.555215} \\
\log \text{ of Ans.} & = 9.365031 \qquad \text{(Art. 106, 4)} \\
\textit{Ans.} & = .231756.
\end{array}$$

Note. A negative quantity can have no common logarithm, as is evident from the definition of Art. 86. If negative quantities occur in computation, they may be treated as if they were positive, and the sign of the result determined irrespective of the logarithmic work.

7. Find the value of $\sqrt[3]{\left(\frac{-.032956}{7.96183}\right)}$.

$$\begin{array}{rl}
\log\sqrt[3]{\left(\frac{.032956}{7.96183}\right)} & = \tfrac{1}{3}(\log .032956 - \log 7.96183) \\
\log .032956 & = 8.517934 \\
\log 7.96183 & = 0.901013 \\
 & 3\underline{)\overline{7.616921}} \\
\log \text{ of Ans.} & = 9.205640 \\
\textit{Ans.} & = -.160561.
\end{array}$$

ARITHMETICAL COMPLEMENT.

108. The *Arithmetical Complement* of the logarithm of a quantity, or, briefly, the *cologarithm* of the quantity, is the logarithm of the reciprocal of that quantity.

For example,

$$\begin{aligned}\text{colog}\,4098 &= \log\frac{1}{4098} = \log 1 - \log 4098\\ &= 0 - 3.612572 = 6.387428 - 10\\ \text{colog}\,.06689 &= \log\frac{1}{.06689} = \log 1 - \log .06689\\ &= 0 - (8.825361 - 10) = 1.174639\end{aligned}$$

The following rule will be evident from the preceding illustrations:

To find the cologarithm of a quantity, subtract its logarithm from 10 – 10.

Note. The cologarithm may be calculated mentally from the logarithm by subtracting the last *significant* figure from 10, and all the others from 9.

The application of this is to exhibit the work of calculation by logarithms in a more compact form in certain cases.

To illustrate, suppose we are to calculate $\frac{a \times b}{c \times d}$ by logarithms.

$$\begin{aligned}\log\left(\frac{a \times b}{c \times d}\right) &= \log\left(a \times b \times \frac{1}{c} \times \frac{1}{d}\right)\\ &= \log a + \log b + \log\frac{1}{c} + \log\frac{1}{d}\\ &= \log a + \log b + \text{colog}\,c + \text{colog}\,d\end{aligned}$$

That is, the work can be exhibited in the form of the addition of four logarithms, instead of the subtraction of the sum of two logarithms from the sum of two others. The principle is only applicable to the case of fractions, and the rule to be used is, *Add together the logarithms of the factors of the numerator and the cologarithms of the factors of the denominator.*

Example. Find the value of $\dfrac{79.23 \times 10.39}{613.8 \times .07723}$.

$$\log \frac{79.23 \times 10.39}{613.8 \times .07723}$$

$$= \log 79.23 + \log 10.39 + \operatorname{colog} 613.8 + \operatorname{colog} .07723.$$

$$\begin{aligned} \log 79.23 &= 1.898890 \\ \log 10.39 &= 1.016616 \\ \operatorname{colog} 613.8 &= 7.211973 - 10 \\ \operatorname{colog} .07723 &= \underline{1.112214} \end{aligned}$$

Adding, $\therefore$ log of Ans. $= 11.239693 - 10 = 1.239693$

Number corresponding $= 17.3657$. *Ans.*

EXAMPLES.

109. Calculate by logarithms the values of the following:

1. $9.23841 \times .00369822$.

2. $\dfrac{3.70963 \times 286.512}{1633.72}$.

3. $(23.8464)^8$.

4. $(-.000929687)^{\frac{2}{7}}$.

5. $\sqrt[8]{3}$.

6. $\sqrt{2}$.

7. $\sqrt[4]{5}$.

8. $\sqrt{.0042937}$.

9. $\sqrt[18]{-6829.586}$.

10. $(1.05624)^{112}$.

11. $(-.00200016)^{\frac{19}{13}}$.

12. $2^{\frac{3}{2}} \times (-3)^{\frac{2}{5}}$.

13. $\dfrac{5^{\frac{3}{7}}}{(-10)^{\frac{2}{9}}}$.

14. $\dfrac{3^{\frac{5}{8}}}{(-4)^{\frac{2}{3}}}$.

15. $\left(\dfrac{6}{7}\right)^{\frac{5}{2}}$.

16. $\sqrt[11]{7239.812}$.

17. $\sqrt[5]{.0001}$.

18. $\sqrt[7]{-.000009506694}$.

19. $\left(\dfrac{35}{113}\right)^{\frac{3}{8}}$.

20. $\left(\frac{.0872635}{.132088}\right)^{\frac{5}{3}}$.

21. $\sqrt[9]{\frac{7}{3}}$.

22. $\sqrt[8]{\frac{21}{13}}$.

23. $\sqrt[5]{\frac{2}{3}} \div \sqrt[8]{\frac{3}{5}}$.

24. $\sqrt[8]{2} \times \sqrt[5]{3} \times \sqrt[7]{.01}$.

25. $\sqrt[5]{\left(\frac{3258.826}{49309.8}\right)}$.

26. $\left(\frac{-31.6259}{429.0162}\right)^{\frac{3}{17}}$.

27. $\frac{100^{\frac{2}{3}}}{(.732465)^{\frac{4}{7}}}$.

28. $\frac{\sqrt[8]{.000128883}}{\sqrt[4]{.000827606}}$.

29. $\frac{(-.746892)^{\frac{5}{3}}}{-(.234521)^{\frac{7}{2}}}$.

30. $\frac{\sqrt[11]{.00730007}}{(.682913)^{\frac{5}{2}}}$.

31. $\frac{\sqrt{5.95463} \times \sqrt[8]{61.1998}}{\sqrt[5]{298.5434}}$.

32. $(538.217 \times .000596899)^{\frac{1}{8}}$.

33. $\frac{-304.698 \times .9026137}{-.00776129 \times -16923.24}$.

34. $(18.9503)^{10} \times (-.1)^{14}$.

35. $\sqrt[6]{3734.89 \times .00001108184}$.

36. $(2.63172)^{\frac{3}{4}} \times (.712719)^{\frac{2}{3}}$.

37. $\frac{\sqrt[8]{-.00819323} \times (.0628513)^{\frac{1}{2}}}{-.9834171}$.

38. $\sqrt{.035} \times \sqrt[6]{.626671} \times \sqrt[3]{.00721033}$.

EXPONENTIAL AND LOGARITHMIC SERIES.

110. We know that for any values of n and x,

$$\left[\left(1+\frac{1}{n}\right)^n\right]^x = \left(1+\frac{1}{n}\right)^{nx}.$$

Expanding by the Binomial Theorem, we obtain

$$\left[1+n\frac{1}{n}+\frac{n(n-1)}{\underline{|2}}\,\frac{1}{n^2}+\frac{n(n-1)(n-2)}{\underline{|3}}\,\frac{1}{n^3}+\cdots\right]^x$$

$$=1+nx\frac{1}{n}+\frac{nx(nx-1)}{\underline{|2}}\,\frac{1}{n^2}+\frac{nx(nx-1)(nx-2)}{\underline{|3}}\,\frac{1}{n^3}+\cdots$$

or,
$$\left[1+1+\frac{1-\frac{1}{n}}{\underline{|2}}+\frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{\underline{|3}}+\cdots\right]^x$$
$$=1+x+\frac{x\left(x-\frac{1}{n}\right)}{\underline{|2}}+\frac{x\left(x-\frac{1}{n}\right)\left(x-\frac{2}{n}\right)}{\underline{|3}}+\cdots.$$

This is true for all values of n; hence, it is true however large n may be. Suppose n to be indefinitely increased. Then the fractions $\frac{1}{n}$, $\frac{2}{n}$, etc., approach the limit 0. Hence, at the limit, we have

$$\left[1+1+\frac{1}{\underline{|2}}+\frac{1}{\underline{|3}}+\cdots\right]^x=1+x+\frac{x^2}{\underline{|2}}+\frac{x^3}{\underline{|3}}+\cdots$$

The series in the bracket we denote by e; hence,

$$e^x=1+x+\frac{x^2}{\underline{|2}}+\frac{x^3}{\underline{|3}}+\cdots$$

111. *To expand a^x in powers of x.*

Let $a=e^m$; whence (Art. 84), $m=\log_e a$.

Then $a^x=e^{mx}=1+mx+\frac{m^2x^2}{\underline{|2}}+\frac{m^3x^3}{\underline{|3}}+\cdots$ (Art. 110)

Substituting the value of m,

$$a^x=1+(\log_e a)\,x+(\log_e a)^2\frac{x^2}{\underline{|2}}+(\log_e a)^3\frac{x^3}{\underline{|3}}+\cdots$$

This result is called the *Exponential Theorem.*

112. The system of logarithms which has e for its base is called the *Napierian System*, from Napier, the inventor of logarithms. The approximate value of e may be easily calculated from the series of Art. 110,

$$e=1+1+\frac{1}{\underline{|2}}+\frac{1}{\underline{|3}}+\frac{1}{\underline{|4}}+\cdots$$

and will be found to be 2.7182818…

113. *To expand* $\log_e(1+x)$ *in powers of* x.

$$a^x = \{1+(a-1)\}^x = 1 + x(a-1) + \frac{x(x-1)}{\underline{|2}}(a-1)^2$$
$$+\frac{x(x-1)(x-2)}{\underline{|3}}(a-1)^3 + \cdots$$
$$= 1 + x\left\{(a-1) - \frac{(a-1)^2}{2} + \frac{(a-1)^3}{3}\cdots\right\} + \text{terms con-}$$

taining x^2, x^3, etc.

But (Art. 111),

$$a^x = 1 + x(\log_e a) + \text{terms containing } x^2, x^3, \text{etc.}$$

As the two values of a^x are equal for all values of x, by the Theorem of Undetermined Coefficients the coefficients of x in the two expressions are equal; hence,

$$\log_e a = (a-1) - \frac{(a-1)^2}{2} + \frac{(a-1)^3}{3}\cdots$$

Putting $a = 1 + x$, and therefore $a - 1 = x$, we obtain

$$\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$$

Note. This formula might be used to calculate Napierian logarithms; but unless x is a very small fraction, the series in the second member is either divergent or converges very slowly, and hence is useless in most cases.

114. *To obtain a more convenient formula for calculating the Napierian logarithm of a number.*

By Art. 113, $\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots$

putting $x = -x$,

$$\log_e(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \cdots$$

Subtracting,

$$\log_e(1+x) - \log_e(1-x) = 2x + \frac{2x^3}{3} + \frac{2x^5}{5} + \cdots$$

or, by Art. 96, $\log_e\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right)$

Let $$x=\frac{1}{2n+1}$$

Then, $$\frac{1+x}{1-x}=\frac{1+\frac{1}{2n+1}}{1-\frac{1}{2n+1}}=\frac{2n+1+1}{2n+1-1}=\frac{n+1}{n}$$

Substituting, $$\log_e\left(\frac{n+1}{n}\right)=\log_e(n+1)-\log_e n$$

$$=2\left(\frac{1}{2n+1}+\frac{1}{3(2n+1)^3}+\frac{1}{5(2n+1)^5}+\cdots\right)$$

That is,

$$\log_e(n+1)=\log_e n+2\left(\frac{1}{2n+1}+\frac{1}{3(2n+1)^3}+\frac{1}{5(2n+1)^5}+\cdots\right)$$

115. To calculate $\log_e 2$, put $n=1$ in the formula of Art. 114.

We then have

$$\log_e 2=\log_e 1+2\left(\frac{1}{2+1}+\frac{1}{3(2+1)^3}+\frac{1}{5(2+1)^5}+\cdots\right)$$

or, since $\log_e 1=0$,

$$\log_e 2=2\left(\frac{1}{3}+\frac{1}{81}+\frac{1}{1215}+\frac{1}{15309}+\frac{1}{177147}+\frac{1}{1948617}+\cdots\right)$$

$$=2\,(.3333333+.0123457+.0008230+.0000653$$
$$+.0000056+.0000005+\cdots)$$

$=2\times .3465734=.6931468=.693147$, correct to the sixth decimal place.

From $\log_e 2$ we may calculate $\log_e 3$; and so on. We shall find $\log_e 10=2.302585$.

116. *To calculate the common logarithm of a number from its Napierian logarithm.*

By Art. 100, changing b to 10, and a to e, we obtain

$$\log_{10}m=\frac{\log_e m}{\log_e 10}=\frac{1}{2.302585}\log_e m=.4342945\times\log_e m.$$

For instance, $\log_{10}2=.4342945\times .693147=.301030$.

The multiplier by which logarithms of any system are derived from the Napierian system, is called the *modulus* of that system. Hence, .4342945 is the modulus of the common system.

As tables of common logarithms are met with more frequently than tables of Napierian, a rule for changing common logarithms into Napierian may be found convenient. We may either divide the common logarithm by the modulus .4342945, or multiply it by 2.302585, the reciprocal of .4342945.

Napierian logarithms are sometimes called *hyperbolic* logarithms, from having been originally derived from the hyperbola. They are also called *natural* logarithms, from being those which occur first in the investigation of a method of calculating logarithms. Napierian logarithms are seldom used in computation, but occur frequently in theoretical investigations.

MISCELLANEOUS EXAMPLES.

117. **1.** Find $\log_3 2187$. (See Art. 84.)

2. Find $\log_5 15625$.

3. Find the logarithm of $\frac{1}{64}$ to the base -2.

4. Find the logarithm of $\frac{1}{32}$ to the base 8.

5. Find the characteristic of $\log_2 183$.

6. Find the characteristic of $\log_5 4203$.

7. Given $\log 2 = .301030$, how many digits are there in 2^{17}?

8. Given $\log 3 = .477121$, how many digits are there in $3^{\frac{55}{4}}$?

9. Find $\log_{12} 56$. (See Art. 100.)

10. Find $\log_8 163$.

11. Find $\log_{20} 411$.

12. Using the table of common logarithms, find the Napierian logarithm of 52.9381. (Art. 116.)

13. Find the Napierian logarithm of 1325.07.

14. Find the Napierian logarithm of .085623.

15. Find the Napierian logarithm of .342977.

VII. CONSTRUCTION AND USE OF TRIGONOMETRICAL TABLES.

118. Trigonometrical Tables are of two kinds; tables of *Natural Functions*, and tables of *Logarithmic Functions*.

A table of natural functions gives the approximate numerical values of the principal functions for angles at regular intervals from 0° to 90°. As a rule, the interval is 1′ or 10″, and the values of the functions are given correct to four, five, or more decimal places.

A table of logarithmic functions gives the logarithms of the corresponding natural functions.

CONSTRUCTION OF A TABLE OF NATURAL SINES AND COSINES.

119. Let it be required to calculate the approximate values of the sines and cosines of angles at intervals of 1′ from 0° to 90°.

We have first to calculate the values of $\sin 1'$ and $\cos 1'$. For the determination of the former we shall make use of the following theorem:

If x is any acute angle expressed in circular measure, then

$$\sin x > x - \frac{x^3}{4}.$$

For, by Art. 67, writing $\frac{x}{2}$ in place of x,

$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$$

Therefore, $$\sin x = 2\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\cos^2\frac{x}{2} = 2\tan\frac{x}{2}\left(1-\sin^2\frac{x}{2}\right)$$

But, by Art. 80, (A), since $\frac{x}{2}$ is acute, $\tan\frac{x}{2} > \frac{x}{2}$. Hence,

$$\sin x > 2\cdot\frac{x}{2}\cdot\left(1-\sin^2\frac{x}{2}\right), \text{ or } > x\left(1-\sin^2\frac{x}{2}\right)$$

But, by Art. 80, (A),

$$\sin\frac{x}{2} < \frac{x}{2}, \text{ or } \sin^2\frac{x}{2} < \frac{x^2}{4}$$

Hence, $$\sin x > x\left(1-\frac{x^2}{4}\right), \text{ or } > x - \frac{x^3}{4}$$

120. By Art. 80, (A), if x is acute, $\sin x < x$; and by Art. 119, $\sin x > x - \frac{x^3}{4}$.

Put x equal to the circular measure of $1' = \frac{\pi}{10800}$, since $1'$ is $\frac{1}{10800}$ of 180° (Art. 6). Taking for π the value 3.14159265, correct to eight decimal places, we have for the circular measure of $1'$ the value .00029088821, correct to eleven decimal places.

Then, $$\sin 1' < .00029088821, \text{ and}$$
$$> .00029088821 - \frac{(.00029088821)^3}{4}$$

The approximate value of the last fraction to eleven decimal places is .00000000001. Thus,

$$\sin 1' > .00029088820.$$

Hence, $\sin 1' = .0002908882$, correct to ten decimal places.

To find the value of $\cos 1'$, we take the formula

$$\begin{aligned}\cos 1' &= \sqrt{1-\sin^2 1'}\\ &= \sqrt{1-(.0002908882)^2}\\ &= .9999999577\end{aligned}$$

121. Having calculated the values of $\sin 1'$ and $\cos 1'$, we obtain the sines and cosines of all angles at intervals of $1'$ from $1'$ to $30°$ as follows:

By Art. 72, $\sin(x+y) = 2\sin x\cos y - \sin(x-y)$

$$\cos(x+y) = 2\cos x\cos y - \cos(x-y)$$

Putting $y = 1'$, these become

$$\sin(x+1') = 2\sin x\cos 1' - \sin(x-1') \qquad \textbf{(59)}$$

$$\cos(x+1') = 2\cos x\cos 1' - \cos(x-1') \qquad \textbf{(60)}$$

Giving to x in these formulæ the value $1'$, we have

$$\begin{aligned}\sin 2' &= 2\sin 1'\cos 1' - \sin 0'\\ &= 2(.0002908882)(.9999999577)\\ &= 2(.0002908882)(1-.0000000423)\\ &= .0005817764\end{aligned}$$

Note. The change in the form of the value of $\cos 1'$ is for convenience in multiplying; it being much less work to multiply .0002908882 by 1 and then by .0000000423 and subtract the second result from the first, than to multiply it at once by .9999999577.

$$\begin{aligned}\cos 2' &= 2\cos 1'\cos 1' - \cos 0' = 2\cos 1'\cos 1' - 1\\ &= 2(.9999999577)(1-.0000000423)-1\\ &= .9999998308\end{aligned}$$

Putting $x = 2'$, we have

$$\begin{aligned}\sin 3' &= 2\sin 2'\cos 1' - \sin 1'\\ &= 2(.0005817764)(1-.0000000423)\\ &\qquad -.0002908882\\ &= .0008726646\end{aligned}$$

$$\begin{aligned}\cos 3' &= 2\cos 2'\cos 1' - \cos 1'\\ &= 2\,(.9999998308)\,(1 - .0000000423) - .9999999577\\ &= .9999996193\end{aligned}$$

By continuing in this way, giving to x the values $3'$, $4'$, $5'$, $\cdots$, $29^\circ\,59'$, we shall have the values of the sines and cosines of all angles at intervals of $1'$ from 0° to 30°.

122. To calculate the values of the sines and cosines for angles from 30° to 45°, we proceed as follows:

By Art. 72, $\sin(x+y) = 2\sin x\cos y - \sin(x-y)$

$$\cos(x+y) = \cos(x-y) - 2\sin x\sin y$$

Putting $x = 30^\circ$, whence $\sin x = \frac{1}{2}$ (Art. 18), these become

$$\sin(30^\circ + y) = \cos y - \sin(30^\circ - y) \qquad (61)$$

$$\cos(30^\circ + y) = \cos(30^\circ - y) - \sin y \qquad (62)$$

Putting $y = 1'$, we have

$$\sin 30^\circ\,1' = \cos 1' - \sin 29^\circ\,59'$$

$$\cos 30^\circ\,1' = \cos 29^\circ\,59' - \sin 1'$$

from which, by aid of the previously calculated values of $\sin 1'$, $\cos 1'$, $\sin 29^\circ\,59'$, and $\cos 29^\circ\,59'$, the sine and cosine of $30^\circ\,1'$ are obtained.

Similarly, giving to y the values $2'$, $3'$, $\cdots$, 15°, we obtain in succession the values of the sines and cosines of all angles at intervals of $1'$ from 30° to 45°.

123. For angles from 45° to 90°, since (Art. 13) the sine and cosine of an angle are respectively the cosine and sine of the complement of the angle, it will be seen that the values already calculated furnish us with the sines and cosines of all angles at intervals of $1'$ from 45° to 90°.

METHODS OF VERIFICATION.

124. In calculating the sine and cosine of $1'$ in Art. 120, we obtained the results correct to ten decimal places; and, in deriving from them the sines and cosines of the remaining angles up to 30°, we performed a great number of operations with these original approximate numbers. It is not likely, then, that all our sines and cosines up to 30° will be obtained correct to ten decimal places, and it is important to know to how many places they may be depended upon as accurate.

It is also important to be able to verify the work from time to time to ascertain if any error has been made in the calculations, since any mistake made in obtaining the value of one of the functions would be repeated to the end of the work.

For this purpose we may compare our results for the angles 15°, 18°, 22° 30′, 30°, 36°, and 45°, calculated in the manner explained, with the sines and cosines of the same angles as derived in Arts. 17, 18, 74, 75, 76, 77. It is also easy, by aid of the formulæ of Art. 69, to derive from the functions of 15° the sines and cosines of the angles 7° 30′, 3° 45′, ⋯; from the functions of 18°, the sines and cosines of 9°, 4° 30′, ⋯; from the functions of 22° 30′, the sines and cosines of 11° 15′, 5° 37′ 30″, ⋯. In this way other portions of the calculated table may be verified.

125. Another method of verification, known as Euler's, is proved as follows:

By Art. 72,

$$\sin(36°+A)-\sin(36°-A)=2\cos 36°\sin A$$
$$=\frac{\sqrt{5}+1}{2}\sin A \qquad \text{(Art. 77)}$$

Also,

$$\sin(72°+A)-\sin(72°-A)=2\cos 72°\sin A$$
$$=\frac{\sqrt{5}-1}{2}\sin A \qquad \text{(Art. 76)}$$

Subtracting these results, we have

$$\sin(36°+A)-\sin(36°-A)-\sin(72°+A)+\sin(72°-A)=\sin A \quad \textbf{(63)}$$

which is *Euler's Formula.*

If to A in this formula we give any value, say 10°, we have

$$\sin 46° - \sin 26° - \sin 82° + \sin 62° = \sin 10°$$

and the calculated values of the sines of these five angles should satisfy this relation if the work has been performed correctly. In this manner we may verify any part of the calculated table of sines.

126. For verifying the table of cosines, we have

$$\begin{aligned} \cos(36° + A) - \cos(36° - A) &= 2\cos 36° \cos A \\ &= \frac{\sqrt{5}+1}{2}\cos A \\ \cos(72° + A) + \cos(72° - A) &= 2\cos 72° \cos A \\ &= \frac{\sqrt{5}-1}{2}\cos A \end{aligned}$$

Subtracting these results,

$$\begin{aligned} \cos(36° + A) + \cos(36° - A) - \cos(72° + A)& \\ - \cos(72° - A) = \cos A& \qquad (64) \end{aligned}$$

This result is known as *Legendre's Formula.*

USE OF THE TABLE OF LOGARITHMIC SINES, COSINES, ETC.

127. This table (Appendix, pages 18 to 61) gives the logarithmic sines, cosines, tangents, and cotangents of all angles at intervals of one minute from 0° to 90°. Since the sine, cosine, tangent, and cotangent of any angle are respectively equal to the cosine, sine, cotangent, and tangent of the complement of the angle, it follows that the functions of angles between 45° and 90° are to be found in the table of functions of angles between 0° and 45°.

For angles between 0° and 45°, the degrees will be found at the *top* of the page, the minutes in the *left hand* column, and the functions in the columns indicated by the names at the *top;* that is, sines in the first column, cosines in the second, tangents in the third, and cotangents in the fourth.

For angles between 45° and 90°, the degrees will be found at the *foot* of the page, the minutes in the *right hand* column, and the functions in the columns indicated by the names at the *foot;* that is, cosines in the first column, sines in the second, cotangents in the third, and tangents in the fourth.

Since the sines and cosines of all angles, the tangents of angles between 0° and 45°, and the cotangents of angles between 45° and 90°, are less than unity, in all these cases the logarithm has been increased by 10, and −10 must be written after the mantissa. The tangents of angles between 45° and 90°, and the cotangents of angles between 0° and 45° being greater than unity, the logarithms are not increased by 10, and their true values are given in the table.

Thus, $\log\sin 38^\circ\, 27' = 9.793673 - 10.$
$\log\tan 66^\circ\, 33' = 0.362735.$
$\log\cot 79^\circ\ \ 8' = 9.283225 - 10.$
$\log\cos 20^\circ\, 54' = 9.970442 - 10.$
$\log\tan 36^\circ\, 12' = 9.864445 - 10.$
$\log\cos 89^\circ\, 37' = 7.825451 - 10.$
$\log\sin 89^\circ\ \ 3' = 9.999940 - 10.$
$\log\cot 12^\circ\, 41' = 0.647713.$

128. *To find the logarithmic sine, cosine, tangent, or cotangent of any acute angle.*

RULE.

Find in the table the logarithmic sine, cosine, tangent, or cotangent of the degrees and minutes.

Find the difference for 1″ *corresponding.* (See Note I. to Art. 128.)

Multiply this difference by the number of seconds.

$\left.\begin{array}{l}\textit{If sine or tangent add}\\ \textit{If cosine or cotangent subtract}\end{array}\right\}$ *this correction.*

Note I. The columns to the right of the columns headed "sine," "cosine," and "tangent," contain the respective differences for one second; the right hand column of differences is also to be used with the column headed "cotangent." It will be observed that the difference does not stand on the same line with the logarithm, but midway between it and the next. When using the degrees at the *top* of the page, the difference next *below* should be taken; with the degrees at the *foot*, the difference next *above*.

Note II. The rule given above assumes that the differences of the logarithmic functions are proportional to the differences of their corresponding angles; which is, in general, unless the angle be very near 0° or 90°, sufficiently exact for practical purposes. Compare Art. 131.

EXAMPLES.

1. Find $\log\tan 17^\circ 13' 51''$.

$$\begin{array}{rl}
 & \log\tan 17^\circ 13' = 9.491180 - 10 \\
 & \qquad\qquad\qquad\quad \underline{379} \\
D.\ 1'' = 7.44 & \qquad\qquad\qquad 9.491559 - 10 \\
\underline{51} & \qquad\qquad\qquad = \log\tan 17^\circ 13' 51''.\ \textit{Ans.} \\
7.44 & \\
\underline{372.0} & \\
379.44 = 379,\ \text{nearly.} &
\end{array}$$

2. Find $\log\cos 66^\circ 33' 29''$.

$$\begin{array}{rl}
 & \log\cos 66^\circ 33' = 9.599827 - 10 \\
 & \qquad\qquad\qquad\quad \underline{141} \\
D.\ 1'' = 4.85 & \qquad\qquad\qquad 9.599686 - 10.\ \textit{Ans.} \\
\underline{29} & \\
43.65 & \\
\underline{97.0} & \\
140.65 = 141,\ \text{nearly.} &
\end{array}$$

Find the values of the following:

3. $\log\cos 35^\circ 18' 42''$.
4. $\log\sin 61^\circ 58' 4''$.
5. $\log\cot 12^\circ 33' 56''$.
6. $\log\cos 0^\circ 47' 38''$.
7. $\log\tan 82^\circ 0' 5.2''$.
8. $\log\sin 55^\circ 11' 43.7''$.
9. $\log\cos 30^\circ 2' 1.83''$.
10. $\log\cot 48^\circ 0' 53''$.

The natural functions of an angle, if required, may be obtained by finding the numbers corresponding to the logarithmic functions.

Thus, in Ex. 1, above, the number corresponding to the logarithm 9.491559 − 10 is .310141, which is the natural tangent of 17° 13′ 51″.

Find the values of the following natural functions:

11. cos 15° 31′ 16″. **13.** cot 50° 12′ 44.2″. .832806

12. tan 72° 0′ 23″. **14.** sin 48° 51′ 6.3″. .75301

129. *To find the acute angle corresponding to a given logarithmic sine, cosine, tangent, or cotangent.*

RULE.

Find in the table the next less logarithmic sine, cosine, tangent, or cotangent, the degrees and minutes corresponding, and the difference for 1″. (See Note I. to Art. 128.)

Subtract the next less from the given logarithm.

Divide the remainder by the difference for 1″, *giving the correction in seconds.*

$$\left.\begin{array}{l}\textit{If sine or tangent add}\\ \textit{If cosine or cotangent subtract}\end{array}\right\}\textit{this correction.}$$

Note I. In searching for the next less logarithm, attention must be paid to the fact that the functions are found in different columns according as the angle is greater or less than 45°.

Suppose now that we are finding the angle corresponding to a logarithmic sine; if we find the next less logarithm in the column headed "sine," we take the degrees from the *top* of the page, and the minutes from the *left hand* column; but if we find it in the column with "sine" at the foot, we take the degrees from the *foot* of the page, and the minutes from the column on the *right.* We proceed similarly with the other three functions.

Note II. It is often convenient, in looking out the angle corresponding to a logarithmic cosine or cotangent, to find the *next greater* logarithm, subtract from it the given logarithm, and *add* the corresponding correction.

EXAMPLES.

1. Find the angle whose log sin = 9.959343 − 10.

$$\begin{array}{rl}
 & 9.959343-10 \\
\text{Next less } \log\sin = & \underline{9.959310-10}\text{; angle cor.} = 65^\circ\, 35'. \\
\text{D. } 1'' = & .95\overline{)\ 33\ (}34.7\text{, nearly} \\
 & \quad\ \ \underline{285} \\
 & \quad\ \ \ \ 450 \\
 & \quad\ \ \ \ \underline{380} \\
 & \quad\ \ \ \ \ \ 700
\end{array}$$

Adding the correction, the angle cor. = 65° 35′ 34.7″. *Ans.*

2. Find the angle whose log cot = 0.169602.

$$\begin{array}{rl}
 & 0.169602 \\
\text{Next less } \log\cot = & \underline{0.169379}\text{; angle cor.} = 34^\circ\, 6'. \\
\text{D. } 1'' = & 4.53\overline{)223(}49.2\text{, nearly.} \\
 & \quad\ \ \underline{1812} \\
 & \quad\ \ \ \ 4180 \\
 & \quad\ \ \ \ \underline{4077} \\
 & \quad\ \ \ \ \ \ 1030
\end{array}$$

Subtracting the correction, the angle cor. = 34° 5′ 10.8″. *Ans.*

Find the angles corresponding in the following:

3. log tan = 0.916351.

4. log cos = 9.923569 − 10.

5. log sin = 9.863011 − 10.

6. log cot = 0.215386.

7. log cos = 9.227937 − 10.

8. log sin = 9.189118 − 10.

If the natural function is given, we have only to find its logarithm and proceed as above.

9. Find the angle whose tangent = .2.

Since log .2 = 9.301030 − 10, we find the angle whose log tan = 9.301030 − 10, which is 11° 18′ 35.8″. *Ans.*

Find the angles corresponding to the following natural functions:

10. $\sin = .096923.$ **12.** $\tan = \frac{2}{3}.$ **14.** $\sin = \frac{1}{8}\sqrt{7}.$

11. $\cot = 2.$ **13.** $\cos = \frac{1}{8}\sqrt{5}.$ **15.** $\tan = .1.$

33°41'.24.3"

130. *To find the logarithmic secant or cosecant of any acute angle.*

Since $$\sec x = \frac{1}{\cos x} \text{ and } \operatorname{cosec} x = \frac{1}{\sin x},$$

we have and $$\left.\begin{aligned} \log \sec x &= \operatorname{colog} \cos x \\ \log \operatorname{cosec} x &= \operatorname{colog} \sin x \end{aligned}\right\} \qquad \text{(Art. 108)}$$

That is,

To find the logarithmic secant, subtract the logarithmic cosine from 10 − 10; *to find the logarithmic cosecant, subtract the logarithmic sine from* 10 − 10.

EXAMPLES.

1. Find log sec 22° 39′.

$$\log \cos 22° \, 39' = 9.965143 - 10.$$

Subtracting from 10 − 10,

$$\log \sec 22° \, 39' = 0.034857. \text{ \textit{Ans.}}$$

Find the values of the following:

2. $\log \operatorname{cosec} 55° \, 11' \, 43''.$ **3.** $\log \sec 80° \, 7' \, 28.2''.$

Find the angles corresponding to the following:

4. $\log \sec = 0.400314.$ **5.** $\log \operatorname{cosec} = 0.188783.$

Note. Since $\cot x = \frac{1}{\tan x}$, we can find the logarithmic cotangent of an angle by subtracting its logarithmic tangent from 10—10.

131. It was stated in Art. 128, Note II., that in general the differences of the logarithmic functions are approximately proportional to the differences of their corresponding angles. It may be seen from the table that this is not the case with the logarithmic sines, tangents, and cotangents of angles near to 0°, nor with the logarithmic cosines, tangents, and cotangents of angles near to 90°.

Thus, according to the table, the difference for 1″ for the sine, tangent, or cotangent of an angle between 1′ and 2′ is 5017.17, while for an angle between 2′ and 3′ it is 2934.85.

Again, the difference for 1″ for the logarithmic sine of an angle near to 90°, or the logarithmic cosine of an angle near to 0°, is so small that it requires a change of many seconds in the angle to produce a change of one unit in the right hand figure of the mantissa.

Thus, according to the table, the angles 1′, 2′, 3′, 4′, and 5′ have the same logarithmic cosine; the reason being that their actual logarithmic cosines do not differ in the first six figures following the decimal point.

132. It follows, then, that the ordinary method cannot be depended upon as accurate in finding the logarithmic sine, tangent, or cotangent of an angle near to 0°, or the logarithmic cosine, tangent, or cotangent of an angle near to 90°; and the same is true in finding the angle corresponding to any logarithmic function when it is near to 0° or 90°.

133. A very accurate method for finding the correction in the case of the logarithmic sine, tangent, or cotangent of a very small angle, is based upon the principle stated in Art. 80, that *the sines and tangents of small angles are approximately proportional to the angles themselves.*

To illustrate, let it be required to find log sin 3′ 18″.

By the principle just stated,

$$\sin 3'\,18'' : \sin 3' = 3'\,18'' : 3' = 198'' : 180''$$

Then, $$\sin 3' 18'' = \frac{198 \sin 3'}{180}$$

or, $$\log \sin 3' 18'' = \log 198 + \log \sin 3' + \text{colog}\, 180$$

$$\begin{aligned} \log 198 &= 2.296665 \\ \log \sin 3' &= 6.940847 - 10 \\ \text{colog}\, 180 &= 7.744727 - 10 \\ \therefore \log \sin 3' 18'' &= 6.982239 - 10 \end{aligned}$$

which is correct to the last figure. By the ordinary method, we should have found the result 6.978329 −10.

Again, let it be required to find the angle corresponding in $\log \sin x = 7.102643 - 10$.

The next less log sine is 7.065786 −10, whose angle corresponding is 4′; then, by the principle enunciated above,

$$\sin x : \sin 4' = x : 4$$

Then, $$x \text{ (in minutes)} = \frac{4 \sin x}{\sin 4'} = 4 \sin x \operatorname{cosec} 4'$$

or, $$\log x = \log 4 + \log \sin x + \log \operatorname{cosec} 4'$$

$$\begin{aligned} \log 4 &= 0.602060 \\ \log \sin x &= 7.102643 - 10 \\ \log \operatorname{cosec} 4' &= 2.934214 \\ \log x &= 0.638917 \\ \therefore x &= 4.35429' = 4' 21.2574'' \end{aligned}$$

The ordinary method gives $x = 4' 22.8''$.

For the logarithmic cotangent of a small angle we may proceed as follows: let it be required to find $\log \cot 2' 11''$.

As above, $$\tan 2' 11'' : \tan 2' = 131 : 120$$

or, $$\tan 2' 11'' = \frac{131 \tan 2'}{120}$$

Consequently (Art. 14), $$\cot 2' 11'' = \frac{120 \cot 2'}{131}$$

To obtain the logarithmic cosine, tangent, or cotangent of an angle near to 90°, we find the logarithmic sine, cotangent, or tangent, respectively, of the complement of the angle.

EXAMPLES.

Find the values of the following:

1. $\log\sin 2' 26''$.
2. $\log\tan 3' 41.8''$.
3. $\log\cot 4' 13''$.
4. $\log\cos 89° 56' 1.2''$.

Find the angles corresponding to the following:

5. $\log\tan = 6.968211 - 10$.
6. $\log\sin = 7.170808 - 10$.
7. $\log\cot = 7.427926 - 10$.

134. By Art. 50, the functions of any angle, positive or negative, can be expressed in terms of the functions of a certain acute angle. Thus the table of the functions of acute angles serves to determine the values of the functions of an angle of any magnitude whatever, positive or negative.

For example, required $\log\sin 172° 15' 11''$.

By Art. 50, $\sin 172° 15' 11'' = \sin(90° + 82° 15' 11'')$
$$= \cos 82° 15' 11''$$

And from the table,

$$\log\cos 82° 15' 11'' = 9.129684 - 10.$$

Or, we may take

$$\sin 172° 15' 11'' = \sin(180° - 7° 44' 49'')$$
$$= \sin 7° 44' 49'' \qquad \text{(Art. 50)}$$

If the natural function is negative, as, for example, in the cosine of an angle between 90° and 180°, there is no logarithmic function, properly speaking. (See Note to Art. 107.) But in the solution of examples in such a case, we may proceed with the function as if it were positive, the algebraic sign of the result being determined irrespective of the logarithmic work. Examples of this will be found in Chapters XV. and XVI.

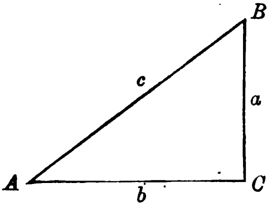
VIII. SOLUTION OF RIGHT TRIANGLES.

135. Every triangle has six elements: three sides and three angles. We know, by Geometry, that a triangle is in general completely determined when three of its six elements are known, provided one of them is a side. The process of finding the unknown from the given elements is called the *solution* of the triangle.

136. For right triangles it is only necessary to know *two* elements in addition to the right angle, one of which must be a side. There will be five cases which can arise:

1. *Given the hypothenuse and an acute angle.*
2. *Given an angle and its opposite side.*
3. *Given an angle and its adjacent side.*
4. *Given the hypothenuse and another side.*
5. *Given the two sides about the right angle.*

137. We shall base the solutions of the different cases under right triangles upon the following set of formulæ:



$$\sin A = \frac{a}{c} \qquad \cos A = \frac{b}{c} \qquad \tan A = \frac{a}{b}$$

$$\sin B = \frac{b}{c} \qquad \cos B = \frac{a}{c} \qquad \tan B = \frac{b}{a}$$

If the two given elements are a side and an angle, the proper formula for computing either of the remaining sides may be found by the following rule:

Take that function of the angle which involves the given side and the required side.

If both the given parts are sides, we should first calculate one of the angles by aid of either formula containing the given elements, and the remaining side may then be found by the above rule.

EXAMPLES.

138. 1. Given $c = 203.762$, $B = 21° 42' 51''$. Find a and b.

In this case the formulæ to be used are

$$\sin B = \frac{b}{c} \text{ and } \cos B = \frac{a}{c}$$

Whence, $$b = c \sin B \text{ and } a = c \cos B$$

By logarithms, $\log b = \log c + \log \sin B$

and $\log a = \log c + \log \cos B$

$$\begin{array}{rlcrl}
\log c = & 2.309123 & \quad & \log c = & 2.309123 \\
\log \sin B = & 9.568174 - 10 & & \log \cos B = & 9.968035 - 10 \\
\cline{2-2} \cline{5-5}
\log b = & 1.877297 & & \log a = & 2.277158 \\
\therefore b = & 75.3871 & & \therefore a = & 189.303
\end{array}$$

2. Given $a = 13$, $A = 67° 7'$. Find b and c.

We have now $\sin A = \dfrac{a}{c}$ and $\tan A = \dfrac{a}{b}$

Whence, $$c = \frac{a}{\sin A} \text{ and } b = \frac{a}{\tan A}$$

By logarithms, $\log c = \log a - \log \sin A$

and $\log b = \log a - \log \tan A$

$$\begin{array}{rlcrl}
\log a = & 1.113943 & \quad & \log a = & 1.113943 \\
\log \sin A = & 9.964400 - 10 & & \log \tan A = & 0.374612 \\
\cline{2-2} \cline{5-5}
\log c = & 1.149543 & & \log b = & 0.739331 \\
\therefore c = & 14.1105 & & \therefore b = & 5.48695
\end{array}$$

3. Given $b = .151213$, $c = .308069$. Find A and a.

We have $\cos A = \frac{b}{c}$; or $\log \cos A = \log b - \log c$.

$$\begin{aligned} \log b &= 9.179589 \\ \log c &= \underline{9.488648} \\ \log \cos A &= 9.690941 \\ \therefore\ A &= 60^\circ\, 36'\, 14.8'' \end{aligned}$$

To find a, we have $\sin A = \frac{a}{c}$, or $a = c \sin A$.

Then, $\log a = \log c + \log \sin A$

$$\begin{aligned} \log c &= 9.488648 \\ \log \sin A &= \underline{9.940142} \\ \log a &= 9.428790 \\ \therefore\ a &= \ .268404 \end{aligned}$$

Note. In this example the —10's are omitted as explained in Art. 91.

139. In the trigonometrical solution of any example where two sides are given, it is necessary to find one of the angles first, and the remaining side may then be calculated. It is possible, however, to determine the remaining side at once, independently of the angle, by Geometrical methods.

Thus, in the example just solved, we have, by Geometry,

$$a^2 + b^2 = c^2;\ \text{whence, } a = \sqrt{c^2 - b^2} = \sqrt{(c+b)(c-b)}.$$

By logarithms, $\log a = \frac{1}{2}\{\log(c+b) + \log(c-b)\}$

From the data, $c + b = .459282$, $c - b = .156856$

$$\begin{aligned} \log(c+b) &= \ 9.662079 - 10 \\ \log(c-b) &= \ 9.195501 - 10 \\ &\ 2\overline{)18.857580 - 20} \\ \log a &= \ 9.428790 - 10 \\ \therefore\ a &= \ .268404, \text{ as before.} \end{aligned}$$

But when the two given sides are a and b, the resulting formula for c is $c = \sqrt{a^2 + b^2}$, which is not adapted to logarithmic computation. In such a case it is shorter to proceed according to the rule of Art. 137.

140. Solve the following triangles:

a 264.952
b 75.9488
1. Given $A = 74° \ 0' \ 18''$, $c = 275.623$.

A 31:52:9.7
B 38:7:50.3
2. Given $a = 638.5$, $b = 501.2$. 811.715 = c

14.3056 = c, 5.41295 = a
3. Given $A = 22° \ 14'$, $b = 13.242$.

c 87.6355
b 26.9148
4. Given $A = 72° \ 6' \ 51''$, $a = 83.4001$.

A 47:51:52.1
b 184.706
5. Given $c = 275.316$, $a = 204.163$.

A 55:43:57.5
c 411.487
6. Given $a = 340.06$, $b = 231.69$.

c 230.535
b 167.593
7. Given $A = 43° \ 22'$, $a = 158.3$.

b 4023.93
a 36987.4
8. Given $B = 6° \ 12' \ 32''$, $c = 37205.62$.

A 36:46:56.7
c 41.2042
9. Given $a = 24.6722$, $b = 33.0011$.

c 53.5747
a 24.3915
10. Given $B = 62° \ 55'$, $b = 47.7$.

A 65:30:18.9
a 3.15341
11. Given $c = 3.465287$, $b = 1.436741$.

A 58:35:43.8
a .0409003
12. Given $c = .04792$, $b = .02497$.

c 456.429
a 410.254
13. Given $A = 64° \ 0' \ 19''$, $b = 200.048$.

b .136736
a .713415
14. Given $B = 10° \ 51'$, $c = .7264$.

c 1876.57
b 76.1761
15. Given $B = 2° \ 19' \ 34''$, $a = 1875.321$.

a 30.5136
b 18.5891
16. Given $A = 58° \ 39'$, $c = 35.73$.

A 30:17:24.4
c 20.0382
17. Given $a = 10.10683$, $b = 17.30264$.

c 213.637
a 176.603
18. Given $B = 34° \ 14' \ 37''$, $b = 120.2162$.

A 58:39:1.8
b 1.04093
19. Given $c = 2.0008$, $a = 1.7087$.

c 7005.71
b 5571.62
20. Given $B = 52° \ 41'$, $a = 4247$.

21. A regular pentagon is inscribed in a circle whose diameter is 24 inches. Find the length of its side. 14.1069"

22. At a distance of 100 feet from the base of a tower, the angle of elevation of its top is observed to be 38°. Find its height. 7[illegible]286

23. The diameter of the earth being taken as 7912 miles, what is the distance of the remotest point of the surface visible from the summit of a mountain $1\frac{1}{4}$ miles in height? 99.4563

24. Find the length of a diagonal of a regular pentagon whose side is 7.028 inches. 11.3715

25. From the top of a lighthouse 133 feet above the sea, the angle of depression of a buoy is observed to be 18° 25′ 6″. Required the horizontal distance of the buoy. 399.386

26. The remotest point of the earth's surface visible from a mountain top is 78.36 miles distant. Find the height of the mountain, the earth's diameter being taken as 7912 miles. .773

27. The length of the side of a regular octagon is 20 inches. Find the radii of the inscribed and circumscribed circles. 26.1313 24.1422

28. How far from the foot of a pole 80 feet high must an observer stand, so that the angle of elevation of the top of the pole may be 10°? 453.702

29. From the top of a tower, the angle of depression of the extremity of a horizontal base line 1000 feet in length, measured from the foot of the tower, is observed to be 21° 16′ 37″. Find the height of the tower. 389.421

30. A regular hexagon is circumscribed about a circle whose diameter is 10 inches. Find the length of its side. 5.77349

141. Care must be taken to employ the more accurate methods of Art. 133 in finding the functions of angles which are near to 0° or 90°, or the angles corresponding under similar circumstances. These rules provide for every case which can arise in the solution of a right triangle, except in looking out the angle corresponding to a logarithmic sine when near to 90°, or a logarithmic cosine when near to 0°.

We will now derive a series of formulæ for right triangles by aid of which the angle corresponding to a logarithmic function may be determined with accuracy when it is near to 90°. An angle near to 0° may always be avoided in the solution of a right triangle by working with the other acute angle.

142. By Art. 11, $\cos A = \frac{b}{c}$.

Hence (Art. 69),

$$2\sin^2 \tfrac{1}{2}A = 1 - \cos A = 1 - \frac{b}{c} = \frac{c-b}{c}$$

and
$$2\cos^2 \tfrac{1}{2}A = 1 + \cos A = 1 + \frac{b}{c} = \frac{c+b}{c}$$

Then,
$$\sin\tfrac{1}{2}A = \sqrt{\frac{c-b}{2c}}, \quad \cos\tfrac{1}{2}A = \sqrt{\frac{c+b}{2c}} \qquad (65)$$

From these, by division,

$$\left.\begin{aligned} \tan\tfrac{1}{2}A &= \sqrt{\frac{c-b}{2c}} \div \sqrt{\frac{c+b}{2c}} = \sqrt{\frac{c-b}{c+b}} \\ \cot\tfrac{1}{2}A &= \sqrt{\frac{c+b}{c-b}} \end{aligned}\right\} \qquad (66)$$

Similarly, we may prove

$$\left.\begin{aligned} \sin\tfrac{1}{2}B &= \sqrt{\frac{c-a}{2c}} \quad & \tan\tfrac{1}{2}B &= \sqrt{\frac{c-a}{c+a}} \\ \cos\tfrac{1}{2}B &= \sqrt{\frac{c+a}{2c}} \quad & \cot\tfrac{1}{2}B &= \sqrt{\frac{c+a}{c-a}} \end{aligned}\right\} \qquad (67)$$

These formulæ involve the *half-angles;* hence, if the angle itself is near to 90°, its half is near to 45°, and the correction in seconds may in that case be found from the table with sufficient precision.

To illustrate, given $c = 1.08261$, $b = 1.08249$; to solve the triangle.

Here A is near to 0° and B is near to 90°, as may be determined by inspection. We should then proceed to find B by the formulæ of Art. 142. For this purpose we must first calculate a, which may be done as in Art. 139.

From the data,

$$c+b=2.1651,\ c-b=.00012.$$

$$\begin{array}{rl}
\log(c+b) = & 0.335478 \\
\log(c-b) = & 6.079181 \\
2\overline{)} & \underline{6.414659} \\
\log a = & 8.207329 \\
\therefore a = & .016119
\end{array}$$

Now, to find B, we use the formula $\sin\frac{1}{2}B=\sqrt{\frac{c-a}{2c}}$.

$$\begin{array}{rlrl}
c-a= & 1.066491 & \log = & 0.027957 \\
2c= & 2.16522 & \log = & 0.335502 \\
 & & 2\overline{)} & \underline{9.692455} \\
 & & \log\sin\frac{1}{2}B = & 9.846227 \\
 & & \therefore \frac{1}{2}B = & 44^\circ\, 34'\, 24.3''
\end{array}$$

Whence, $B=89^\circ\, 8'\, 48.6''$, and $A=0^\circ\, 51'\, 11.4''$.

If b is small compared with c, then A is near to 90°, and should be calculated directly by aid of the formulæ of this article.

If a and b are the given sides, and b is very small compared with a, then A is near to 90°, and should be calculated by the formulæ of this article, c being first computed by the formula $c=\sqrt{a^2+b^2}$, as in Art. 139. If b is very large compared with a, then B is near to 90°, and should be the first element determined in the computation.

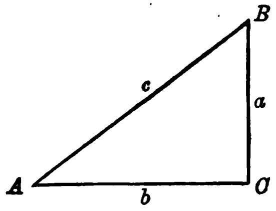
EXAMPLES.

Find the angles in the following right triangles:

1. Given $a=.0128$, $c=152.337$.
2. Given $b=5.81006$, $c=5.81039$.
3. Given $c=11527.2$, $b=1.32$.
4. Given $a=.77$, $c=98276.4$.
5. Given $a=42.0098$, $c=42.0103$.

FORMULÆ FOR THE AREA OF A RIGHT TRIANGLE.

143. Case I. *Given the hypothenuse and an acute angle.*



Letting K denote the area, we have, by Geometry,

$$2K = ab$$

But (Art. 12), $a = c\sin A$, $b = c\cos A$

Substituting, $2K = c^2\sin A\cos A = \frac{1}{2}c^2\sin 2A$ (Art. 67)

Whence, $$4K = c^2\sin 2A \tag{68}$$

Similarly, we may prove

$$4K = c^2\sin 2B \tag{69}$$

Case II. *Given an angle and its opposite side.*

Since $b = \frac{a}{\tan A}$ (Art. 11), we have

$$2K = a \cdot \frac{a}{\tan A} = a^2\cot A \tag{70}$$

Similarly, $$2K = b^2\cot B \tag{71}$$

Case III. *Given an angle and its adjacent side.*

Since $b = a\tan B$ (Art. 11), we have

$$2K = a \cdot a\tan B = a^2\tan B \tag{72}$$

Similarly, $$2K = b^2\tan A \tag{73}$$

Case IV. *Given the hypothenuse and another side.*

Since $b = \sqrt{(c+a)(c-a)}$ (Art. 139), we have

$$2K = a\sqrt{(c+a)(c-a)} \qquad (74)$$

Similarly, $$2K = b\sqrt{(c+b)(c-b)} \qquad (75)$$

Case V. *Given the two sides about the right angle.*

Here, $$2K = ab \qquad (76)$$

EXAMPLES.

1. Given $c = 10.3572$, $B = 15° 2' 46''$; find the area.

By Case I., $$4K = c^2 \sin 2B$$

Hence, $$\log(4K) = 2\log c + \log\sin 2B$$

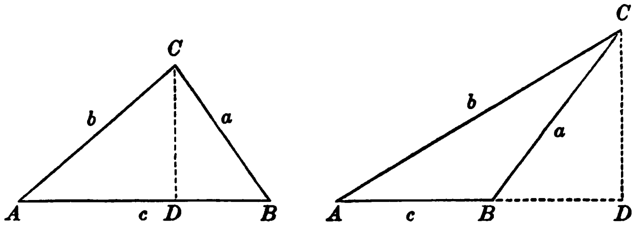
$$\begin{array}{ll} \log c = 1.015242 & \therefore 2\log c = 2.030484 \\ 2B = 30° 5' 32'' & \log\sin = \underline{9.700178 - 10} \\ & \log(4K) = 1.730662 \\ & \therefore 4K = 53.7851 \\ & \therefore K = 13.4463 \end{array}$$

Find the area in each of the following:

2. Given $A = 18° 16' 39''$, $a = 203.078$. 62432.5
3. Given $B = 21° 33' 51''$, $a = .8210733$.
4. Given $a = 100.58$, $b = 115.738$.
5. Given $b = 2.887262$, $c = 5.110177$.
6. Given $A = 43° 28' 3''$, $c = 11.229$.
7. Given $B = 72° 8' 16''$, $c = .0272768$.
8. Given $A = 68° 48' 42''$, $b = 729.344$.
9. Given $a = .62408$, $c = .90983$.
10. Given $B = 30° 30' 57''$, $b = 1.643829$.

IX. GENERAL PROPERTIES OF TRIANGLES.

144. *In any triangle the sides are proportional to the sines of their opposite angles.*



There will be two cases which can arise, according as the angles are all acute, or one of them obtuse. In each figure let the line CD be drawn perpendicular to AB.

From the left hand figure we have, by Art. 12,

$$CD = b \sin A \text{ and } CD = a \sin B$$

That is, $\quad b \sin A = a \sin B$

whence, by the theory of proportion,

$$a : b = \sin A : \sin B \qquad (77)$$

From the right hand figure, by Art. 12,

$$CD = b \sin A \text{ and } CD = a \sin CBD$$

But $\quad \sin CBD = \sin (180° - B) = \sin B \qquad$ (Art. 43)

Then, as before, $b \sin A = a \sin B$

or, $\quad a : b = \sin A : \sin B$

In a similar manner we may prove

$$b : c = \sin B : \sin C \qquad (78)$$

$$c : a = \sin C : \sin A \qquad (79)$$

These results may be expressed more compactly as follows:

$$a : b : c = \sin A : \sin B : \sin C$$

or,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

145. *In any triangle the sum of any two sides is to their difference as the tangent of half the sum of the opposite angles is to the tangent of half their difference.*

By Art. 144, $a : b = \sin A : \sin B$

Whence, by the theory of proportion,

$$a + b : a - b = \sin A + \sin B : \sin A - \sin B$$

or,

$$\frac{a+b}{a-b} = \frac{\sin A + \sin B}{\sin A - \sin B}$$

But, by Art. 73,

$$\frac{\sin A + \sin B}{\sin A - \sin B} = \frac{\tan \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)}$$

Hence,

$$\frac{a+b}{a-b} = \frac{\tan \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)} \tag{80}$$

Similarly, we may prove

$$\frac{b+c}{b-c} = \frac{\tan \frac{1}{2}(B+C)}{\tan \frac{1}{2}(B-C)} \tag{81}$$

$$\frac{c+a}{c-a} = \frac{\tan \frac{1}{2}(C+A)}{\tan \frac{1}{2}(C-A)} \tag{82}$$

Since

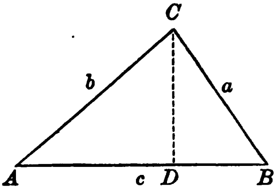
$$\begin{aligned}\tan \tfrac{1}{2}(A+B) &= \tan \tfrac{1}{2}(180° - C) \\ &= \tan(90° - \tfrac{1}{2}C) = \cot \tfrac{1}{2}C\end{aligned}$$

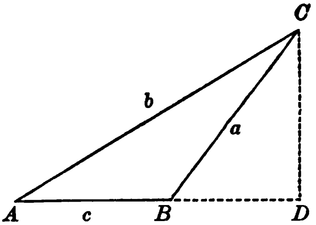
the result (80) may also be written

$$\frac{a+b}{a-b} = \frac{\cot \frac{1}{2}C}{\tan \frac{1}{2}(A-B)} \tag{83}$$

146. *In any triangle the square of any side is equal to the sum of the squares of the other two sides, minus twice their product into the cosine of their included angle.*

Case I. *When the included angle is acute.*





There will be two cases to consider according as the angles are all acute, or one of them obtuse. In each figure let CD be drawn perpendicular to AB.

From the left hand figure,

$$BD = c - AD$$

From the right hand figure,

$$BD = AD - c$$

Squaring, in either case,

$$BD^2 = AD^2 + c^2 - 2c \times AD$$

Adding CD^2 to both members,

$$BD^2 + CD^2 = AD^2 + CD^2 + c^2 - 2c \times AD$$

But, by Geometry,

$$BD^2 + CD^2 = a^2, \text{ and } AD^2 + CD^2 = b^2$$

Also (Art. 12), $AD = b \cos A$

Hence, $a^2 = b^2 + c^2 - 2bc \cos A$

Case II. *When the included angle is obtuse.*

From the right hand figure above,

$$AD = c + BD$$

Squaring, and adding CD^2 to both members,

$$AD^2+CD^2=c^2+BD^2+CD^2+2c\times BD$$

But, $AD^2+CD^2=b^2$, and $BD^2+CD^2=a^2$

Also (Art. 43),

$$\cos B=\cos(180^\circ-CBD)$$

$$=-\cos CBD=-\frac{BD}{a}$$

That is, $BD=-a\cos B$

Hence, $b^2=c^2+a^2-2ca\cos B$

We then have the three formulæ,

$$a^2=b^2+c^2-2bc\cos A \qquad (84)$$

$$b^2=c^2+a^2-2ca\cos B \qquad (85)$$

$$c^2=a^2+b^2-2ab\cos C \qquad (86)$$

147. *To express the cosines of the angles of a triangle in terms of the sides.*

By Art. 146, $a^2=b^2+c^2-2bc\cos A$

Whence, $2bc\cos A=b^2+c^2-a^2$

or, $$\cos A=\frac{b^2+c^2-a^2}{2bc} \qquad (87)$$

Similarly, we have,

$$\cos B=\frac{c^2+a^2-b^2}{2ca}, \qquad \cos C=\frac{a^2+b^2-c^2}{2ab} \qquad (88)$$

148. *To express the sines, cosines, and tangents of the half-angles of a triangle in terms of the sides.*

By Art. 147, $\cos A=\frac{b^2+c^2-a^2}{2bc}$ (A)

Subtracting both members from unity,

$$1-\cos A=1-\frac{b^2+c^2-a^2}{2bc}=\frac{2bc-b^2-c^2+a^2}{2bc}$$

That is, $$2\sin^2\tfrac{1}{2}A=\frac{a^2-(b-c)^2}{2bc} \qquad \text{(Art. 69)}$$

or, $$\sin^2\tfrac{1}{2}A=\frac{(a+b-c)(a-b+c)}{4bc}$$

Let $a+b+c$ be represented by $2s$, s being then the half-sum of the sides of the triangle. Then,

$$a+b-c=a+b+c-2c=2s-2c=2(s-c)$$
$$a-b+c=a+b+c-2b=2s-2b=2(s-b)$$

Substituting, $$\sin^2\tfrac{1}{2}A=\frac{4(s-b)(s-c)}{4bc}$$

or, $$\sin\tfrac{1}{2}A=\sqrt{\frac{(s-b)(s-c)}{bc}} \qquad \textbf{(89)}$$

Similarly, $$\sin\tfrac{1}{2}B=\sqrt{\frac{(s-c)(s-a)}{ca}} \qquad \textbf{(90)}$$

$$\sin\tfrac{1}{2}C=\sqrt{\frac{(s-a)(s-b)}{ab}} \qquad \textbf{(91)}$$

Again, adding both members of (**A**) to unity, we have

$$1+\cos A=1+\frac{b^2+c^2-a^2}{2bc}$$
$$=\frac{2bc+b^2+c^2-a^2}{2bc}$$

That is, $$2\cos^2\tfrac{1}{2}A=\frac{(b+c)^2-a^2}{2bc} \qquad \text{(Art. 69)}$$

or, $$\cos^2\tfrac{1}{2}A=\frac{(b+c+a)(b+c-a)}{4bc}$$

But, $b+c+a=2s$

and $b+c-a=b+c+a-2a=2(s-a)$

Hence, $$\cos^2\tfrac{1}{2}A=\frac{4s(s-a)}{4bc}$$

or, $$\cos\tfrac{1}{2}A=\sqrt{\frac{s(s-a)}{bc}} \qquad \textbf{(92)}$$

Similarly,
$$\cos\tfrac{1}{2}B=\sqrt{\frac{s(s-b)}{ca}} \tag{93}$$

$$\cos\tfrac{1}{2}C=\sqrt{\frac{s(s-c)}{ab}} \tag{94}$$

Dividing (89) by (92), we obtain

$$\tan\tfrac{1}{2}A=\sqrt{\frac{(s-b)(s-c)}{bc}}\sqrt{\frac{bc}{s(s-a)}}$$

$$=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}} \tag{95}$$

Similarly, we may prove

$$\tan\tfrac{1}{2}B=\sqrt{\frac{(s-c)(s-a)}{s(s-b)}} \tag{96}$$

$$\tan\tfrac{1}{2}C=\sqrt{\frac{(s-a)(s-b)}{s(s-c)}} \tag{97}$$

Since the angles of a triangle are each less than 180°, their halves are less than 90°; hence, the *positive* sign must be taken before the radicals in the formulæ of this article.

149. *To express the sine of either angle of a triangle in terms of the sides.*

By Art. 67, $\sin A=2\sin\tfrac{1}{2}A\cos\tfrac{1}{2}A$

$$=2\sqrt{\frac{(s-b)(s-c)}{bc}}\sqrt{\frac{s(s-a)}{bc}} \quad \text{(Art. 148)}$$

$$=\frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)} \tag{98}$$

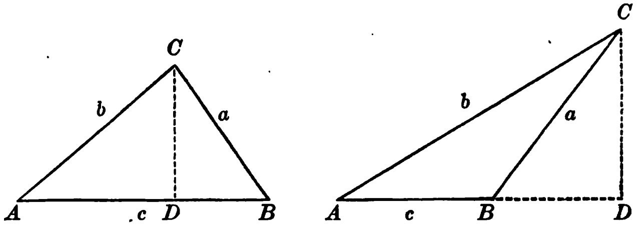
Similarly, we may prove

$$\sin B=\frac{2}{ca}\sqrt{s(s-a)(s-b)(s-c)} \tag{99}$$

$$\sin C=\frac{2}{ab}\sqrt{s(s-a)(s-b)(s-c)} \tag{100}$$

FORMULÆ FOR THE AREA OF AN OBLIQUE TRIANGLE.

150. Case I. *Given two sides and their included angle.*



We have two cases according as the angles are all acute, or one obtuse. Draw in each figure CD perpendicular to AB. Let K denote the area. Then, by Geometry,

$$2K = c \times CD$$

But, in either figure, by Art. 12, $CD = b \sin A$.

Hence, $$2K = bc \sin A \qquad (101)$$

Similarly, $$2K = ca \sin B \qquad (102)$$

$$2K = ab \sin C \qquad (103)$$

Case II. *Given one side and the angles.*

By Case I., $$2K = ab \sin C$$

But, by Art. 144, $\dfrac{b}{\sin B} = \dfrac{a}{\sin A}$, or $b = \dfrac{a \sin B}{\sin A}$

Substituting, $$2K = a \cdot \frac{a \sin B}{\sin A} \cdot \sin C$$

$$= \frac{a^2 \sin B \sin C}{\sin A} \qquad (104)$$

Similarly, $$2K = \frac{b^2 \sin C \sin A}{\sin B} \qquad (105)$$

$$2K = \frac{c^2 \sin A \sin B}{\sin C} \qquad (106)$$

Case III. *Given the three sides.*

By Case I., $2K = bc\sin A$

But, by Art. 149,

$$\sin A = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}$$

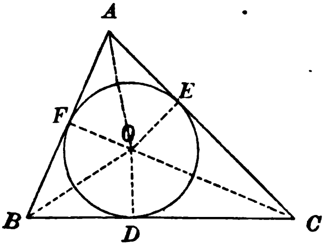
Substituting, $2K = 2\sqrt{s(s-a)(s-b)(s-c)}$

or, $$K = \sqrt{s(s-a)(s-b)(s-c)} \qquad (107)$$

INSCRIBED, ESCRIBED, AND CIRCUMSCRIBED CIRCLES.

151. *To find an expression for the radius of the inscribed circle of a triangle.*

Let ABC be a triangle, and O the centre of its inscribed circle. Draw the radii OD, OE, OF to the points of contact, and join OA, OB, OC.



Since, by Geometry, OD is perpendicular to BC, OE to AC, and OF to AB, we have, denoting the radius of the circle by r,

$$\text{Area } OBC = \tfrac{1}{2}OD \times BC = \tfrac{1}{2}ra$$
$$\text{Area } OCA = \tfrac{1}{2}OE \times CA = \tfrac{1}{2}rb$$
$$\text{Area } OAB = \tfrac{1}{2}OF \times AB = \tfrac{1}{2}rc$$

Adding, $\therefore$ Area $ABC = \tfrac{1}{2}r(a+b+c)$

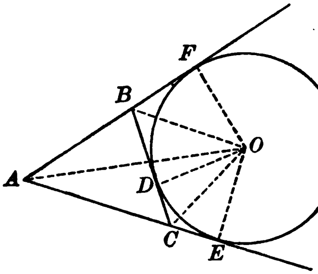
Or, with the notation of Arts. 148 and 150,

$$K = rs$$

Hence, $$r = \frac{K}{s} \qquad (108)$$

That is, *the radius of the inscribed circle is equal to the area of the triangle divided by its semi-perimeter.*

152. *To find expressions for the radii of the escribed circles of a triangle.*



Let ABC be any triangle, and O the centre of the escribed circle touching the sides BC, and AB and AC produced. Draw the radii OD, OE, OF to the points of contact, and join OA, OB, OC. Denoting the radius of the circle by r_1, we have

$$\text{Area } OBC = \tfrac{1}{2} OD \times BC = \tfrac{1}{2} r_1 a$$

$$\text{Area } OCA = \tfrac{1}{2} OE \times CA = \tfrac{1}{2} r_1 b$$

$$\text{Area } OAB = \tfrac{1}{2} OF \times AB = \tfrac{1}{2} r_1 c$$

But, Area $ABC =$ Area $OCA +$ Area $OAB -$ Area OBC

Hence, $$K = \tfrac{1}{2} r_1 (b + c - a)$$

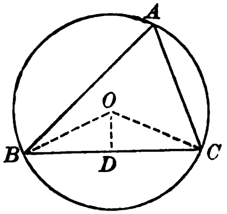
But, $\tfrac{1}{2}(b + c - a) = s - a$ (Art. 148)

Hence, $$K = r_1(s - a), \text{ or } r_1 = \frac{K}{s - a} \qquad (109)$$

Similarly, if r_2 be the radius of the escribed circle touching AC, and AB and BC produced, and r_3 the radius of the circle touching AB, and AC and BC produced, we shall have

$$r_2 = \frac{K}{s - b}, \qquad r_3 = \frac{K}{s - c} \qquad (110)$$

153. *To find an expression for the radius of the circumscribed circle of a triangle.*



Let ABC be any triangle, and O the centre of its circumscribed circle. Draw the radii OB, OC, and draw OD perpendicular to BC. Denoting the radius of the circle by R, we have

$$\sin BOD = \frac{BD}{OB} = \frac{BD}{R}$$

But, by Geometry, the angle $BOC = 2A$; hence, $BOD = \frac{1}{2}BOC = A$. Also, $BD = \frac{1}{2}BC = \frac{1}{2}a$. Hence,

$$\sin A = \frac{\frac{1}{2}a}{R}, \text{ or } R = \frac{a}{2\sin A} \qquad (111)$$

Similarly, we may prove

$$R = \frac{b}{2\sin B}, \quad R = \frac{c}{2\sin C} \qquad (112)$$

154. By Art. 150, $2K = bc\sin A$; hence, $\frac{1}{\sin A} = \frac{bc}{2K}$; substituting in (111), we have

$$R = \frac{a}{2} \cdot \frac{bc}{2K} = \frac{abc}{4K} \qquad (113)$$

That is, *the radius of the circumscribed circle of a triangle is equal to the product of the sides divided by four times the area.*

EXERCISES.

155. 1. In any right triangle, $\cot\frac{A}{2} = \frac{b+c}{a}$.

Prove the following theorems for any triangle:

2. $a\cos B + b\cos A = c.$

3. $a\cos B - b\cos A = \dfrac{a^2 - b^2}{c}.$

4. $\tan B = \dfrac{b\sin C}{a - b\cos C}.$

5. $\left\{\dfrac{\sin\frac{1}{2}A}{\sin\frac{1}{2}B}\right\}^2 = \dfrac{a(s-b)}{b(s-a)}.$

6. $bc\cos A + ca\cos B + ab\cos C = \frac{1}{2}(a^2 + b^2 + c^2).$

7. $\cot\frac{1}{2}A = \dfrac{b+c-a}{2a}(\cot\frac{1}{2}B + \cot\frac{1}{2}C).$

8. $(b+c)^2\sin^2\frac{1}{2}A + (b-c)^2\cos^2\frac{1}{2}A = a^2.$

9. $\dfrac{b\cos B + c\cos C}{a} = \cos(B - C).$

10. $(b+c)\cos A + (c+a)\cos B + (a+b)\cos C =$ the perimeter.

11. $a\cos A + b\cos B + c\cos C = 2a\sin B\sin C.$

12. $K = \dfrac{a^2\sin 2B + b^2\sin 2A}{4}.$

13. $2Rr = \dfrac{abc}{a+b+c}.$

14. $(s-a)\tan\frac{1}{2}A = r.$

15. $rr_1r_2r_3 = K^2.$

16. $\sin^2\frac{1}{2}A = \dfrac{K^2}{bcr_2r_3}.$

17. $\dfrac{1}{r} = \dfrac{1}{r_1} + \dfrac{1}{r_2} + \dfrac{1}{r_3}.$

18. $\sin\frac{1}{2}A\sin\frac{1}{2}B\sin\frac{1}{2}C = \dfrac{K^2}{abcs}.$

19. $\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C = \dfrac{s}{4R}.$

20. $r = \sqrt{\dfrac{(s-a)(s-b)(s-c)}{s}}.$

X. SOLUTION OF OBLIQUE TRIANGLES.

In the solution of plane oblique triangles we may distinguish four cases.

156. CASE I. *Given a side and any two angles.*

We first find the other angle, by Geometry; then, by the aid of Art. 144, the remaining sides may be calculated. The triangle is possible for any values of the data, provided the sum of the given angles is less than 180°.

Example. Given $b = 20.2367$, $B = 19^\circ\,21'\,6''$, $A = 103^\circ\,35'\,52''$. Find C, a, and c.

$$C = 180^\circ - (A + B) = 180^\circ - 122^\circ\,56'\,58'' = 57^\circ\,3'\,2''$$

By Art. 144, $\dfrac{a}{b} = \dfrac{\sin A}{\sin B}$ and $\dfrac{c}{b} = \dfrac{\sin C}{\sin B}$

That is, $a = b \sin A \operatorname{cosec} B$, $c = b \sin C \operatorname{cosec} B$

By logarithms,

$$\log a = \log b + \log \sin A + \log \operatorname{cosec} B$$
$$\log c = \log b + \log \sin C + \log \operatorname{cosec} B$$

$$\begin{array}{rlcrl}
\log b = & 1.306140 & & \log b = & 1.306140 \\
\log \sin A = & 9.987652 - 10 & & \log \sin C = & 9.923840 - 10 \\
\log \operatorname{cosec} B = & 0.479693 & & \log \operatorname{cosec} B = & 0.479693 \\
\hline
\log a = & 1.773485 & & \log c = & 1.709673 \\
\therefore a = & 59.3588 & & \therefore c = & 51.2475
\end{array}$$

Note. In finding $\log \sin 103^\circ\,35'\,52''$, we may take either $\log \cos 13^\circ\,35'\,52''$ or $\log \sin 76^\circ\,24'\,8''$. (See Art. 134.)

To find the log cosecant of any angle, use the rule of Art. 130.

157. CASE II. *Given two sides and their included angle.*

Since one angle is known, the sum of the other two angles is known; and their difference may be calculated by aid of the formulæ of Art. 145. Then the angles themselves may be found. Finally, we may obtain the remaining side as in

Case I. The triangle is always possible for any values of the given parts.

Example. Given $a = 167$, $c = 82$, $B = 98^\circ\, 14'$. Find A, C, and b.

We have

$$A + C = 180^\circ - B = 81^\circ\, 46'; \text{ or, } \tfrac{1}{2}(A + C) = 40^\circ\, 53'$$

By Art. 145, $$\frac{a+c}{a-c} = \frac{\tan\frac{1}{2}(A+C)}{\tan\frac{1}{2}(A-C)}$$

or, $$\tan\tfrac{1}{2}(A-C) = \frac{a-c}{a+c}\tan\tfrac{1}{2}(A+C)$$

By logarithms,

$$\log\tan\tfrac{1}{2}(A-C) = \log(a-c) + \operatorname{colog}(a+c) + \log\tan\tfrac{1}{2}(A+C)$$

$$\begin{array}{lr} a - c = 85 & \log = 1.929419 \\ a + c = 249 & \operatorname{colog} = 7.603801 \\ \tfrac{1}{2}(A+C) = 40^\circ\, 53' & \log\tan = \underline{9.937376} \\ & \log\tan\tfrac{1}{2}(A-C) = 9.470596 \\ & \therefore \tfrac{1}{2}(A-C) = 16^\circ\, 27'\, 49.7'' \end{array}$$

Hence, $$A = \tfrac{1}{2}(A+C) + \tfrac{1}{2}(A-C) = 57^\circ\, 20'\, 49.7''$$
$$C = \tfrac{1}{2}(A+C) - \tfrac{1}{2}(A-C) = 24^\circ\, 25'\, 10.3''$$

For the remaining side,

$$b = \frac{a\sin B}{\sin A} = a\sin B\operatorname{cosec} A \qquad \text{(Art. 144)}$$

or, $$\log b = \log a + \log\sin B + \log\operatorname{cosec} A$$

$$\begin{aligned} \log a &= 2.222716 \\ \log\sin B &= 9.995501 \\ \log\operatorname{cosec} A &= \underline{0.074711} \\ \log b &= 2.292928 \\ \therefore b &= 196.304 \end{aligned}$$

158. In obtaining the remaining side by the method of the preceding article, three new logarithms are required; but it is possible to determine it by another method, in which only two new logarithms are required.

By Art. 144, $$a \sin B = b \sin A$$

and, $$a \sin C = c \sin A$$

Adding, $$a(\sin B + \sin C) = (b + c)\sin A \qquad (\text{A})$$

By Art. 72, $$\sin B + \sin C = 2\sin\tfrac{1}{2}(B + C)\cos\tfrac{1}{2}(B - C)$$

But $$B + C = 180^\circ - A$$

hence, $$\sin\tfrac{1}{2}(B + C) = \sin(90^\circ - \tfrac{1}{2}A) = \cos\tfrac{1}{2}A$$

Also (Art. 67), $$\sin A = 2\sin\tfrac{1}{2}A\cos\tfrac{1}{2}A$$

Substituting in (A),

$$2a\cos\tfrac{1}{2}A\cos\tfrac{1}{2}(B - C) = 2(b + c)\sin\tfrac{1}{2}A\cos\tfrac{1}{2}A$$

Whence, $$a = \frac{(b + c)\sin\frac{1}{2}A}{\cos\frac{1}{2}(B - C)} = (b + c)\sin\tfrac{1}{2}A\sec\tfrac{1}{2}(B - C)$$

Similarly, we may prove

$$b = (c + a)\sin\tfrac{1}{2}B\sec\tfrac{1}{2}(C - A)$$

$$c = (a + b)\sin\tfrac{1}{2}C\sec\tfrac{1}{2}(A - B)$$

To determine b by this method in the example of Art. 157, we have

$$\log b = \log(c + a) + \log\sin\tfrac{1}{2}B + \log\sec\tfrac{1}{2}(A - C)$$

$$\begin{array}{rlrl}
c + a = & 249 & \log = & 2.396199 \\
\frac{1}{2}B = & 49^\circ\ 7' & \log\sin = & 9.878547 \\
\frac{1}{2}(A - C) = & 16^\circ\ 27'\ 49.7'' & \log\sec = & \underline{0.018182} \\
 & & \log b = & 2.292928 \\
 & & \therefore b = & 196.304, \text{ as before.}
\end{array}$$

By this method the only new logarithms are those of $\sin\frac{1}{2}B$ and $\cos\frac{1}{2}(C - A)$; the other logarithm, that of $c + a$, being used in the former computation.

159. CASE III. *Given two sides and the angle opposite to one of them.*

It was stated in Art. 135 that a triangle is in general completely determined when three of its six elements are given, provided one of them is a side. The only exceptions occur when the given parts are those of Case III.

To illustrate, let us consider the following:

Example. Given $a = 52.1$, $b = 61.2$, $A = 31° 26'$; to find c, B, and C.

By Art. 144, $$\frac{\sin B}{\sin A} = \frac{b}{a}$$

or, $$\sin B = \frac{b \sin A}{a}$$

That is, $$\log \sin B = \log b + \operatorname{colog} a + \log \sin A$$

$$\begin{aligned} \log b &= 1.786751 \\ \operatorname{colog} a &= 8.283162 \\ \log \sin A &= 9.717259 \\ \hline \log \sin B &= 9.787172 \end{aligned}$$

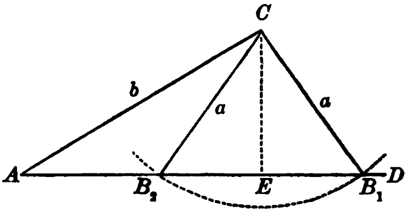
$\therefore B = 37° 46' 37.9''$, from the table.

But, in looking out the angle corresponding, attention must be paid to the fact that an angle and its supplement have the same sine (Art. 43). Thus another value of B will be the supplement of $37° 46' 37.9''$, or $142° 13' 22.1''$. Calling these values B_1 and B_2, we have

$$B_1 = 37° 46' 37.9'', \quad B_2 = 142° 13' 22.1''.$$

In every case where an angle of an oblique triangle has to be determined from its sine this ambiguity occurs, and both values must be retained in the answer, unless one of them can be shown by other considerations to be inapplicable.

The cause of the ambiguity is at once apparent when we attempt to plot the triangle from the data.



We first lay off AC or b, 61.2 units in length, and draw the indefinite straight line AD, making the angle A equal to 31° 26′. Then, with C as a centre and a radius equal to a, that is, 52.1 units, we describe an arc of a circle, cutting AD at B_1 and B_2. Then either of the triangles, AB_1C or AB_2C, satisfies the given conditions.

We now see the reason why an example under Case III. generally gives two solutions. The values of B obtained are the magnitudes of the angles CB_1A and CB_2A, respectively; and it is evident, geometrically, that these two angles are supplements of each other.

To complete the solution, let us denote the angles ACB_1 and ACB_2 by C_1 and C_2, respectively, and the sides AB_1 and AB_2 by c_1 and c_2. Then,

$$C_1 = 180° - (A + B_1) = 180° - 69° 12' 37.9'' = 110° 47' 22.1''$$

$$C_2 = 180° - (A + B_2) = 180° - 173° 39' 22.1'' = 6° 20' 37.9''$$

Again, $c_1 = \dfrac{a \sin C_1}{\sin A}$ and $c_2 = \dfrac{a \sin C_2}{\sin A}$ (Art. 144)

or, $c_1 = a \sin C_1 \operatorname{cosec} A, \quad c_2 = a \sin C_2 \operatorname{cosec} A$

That is,

$$\log c_1 = \log a + \log \sin C_1 + \log \operatorname{cosec} A$$

$$\log c_2 = \log a + \log \sin C_2 + \log \operatorname{cosec} A$$

$$\begin{array}{rlcrl}
\log a = & 1.716838 & \qquad & \log a = & 1.716838 \\
\log \sin C_1 = & 9.970761 & & \log \sin C_2 = & 9.043343 \\
\log \operatorname{cosec} A = & 0.282741 & & \log \operatorname{cosec} A = & 0.282741 \\
\hline
\log c_1 = & 1.970340 & & \log c_2 = & 1.042922 \\
\therefore c_1 = & 93.3985 & & \therefore c_2 = & 11.0388
\end{array}$$

160. A consideration of the different cases which can arise in plotting a triangle, when two sides and an angle opposite to one of them are given, will afford a means of determining when an example under Case III. will have two solutions, when only one, and when none.

If either of the points B_1 and B_2 falls to the *left* of A in the line AD, the triangle corresponding could not be regarded as answering to the conditions, for it would not contain the given angle A. If either B_1 or B_2 coincides with A, one of the triangles will, in that case, reduce to a straight line, which is not considered as a solution. If the circular arc meets the line AD in *one point* only, that is, if it is tangent to it, we shall have only one triangle, which will be right angled; and, finally, if the arc does not meet AD at all, the triangle will be impossible.

The cases which can arise, and the corresponding results, are enumerated below. By Art. 12, the length of the perpendicular CE, from C to AD, is $b \sin A$. Then,

If A is acute, and $a > b$, B_2 falls to the left of A, and there is but one solution.

If A is acute, and $a = b$, B_2 coincides with A, and there is but one solution; that is, an isosceles triangle.

If A is acute, $a < b$, and $b \sin A < a$, B_1 and B_2 fall to the right of A, and there are two solutions.

If A is acute, $a < b$, and $b \sin A = a$, B_1 and B_2 coincide at E, and there is but one solution; that is, a right triangle.

If A is acute, $a < b$, and $b \sin A > a$, the circular arc does not meet AD at all, and there is no solution.

If A is right or obtuse, and $a > b$, B_1 falls to the right of A, and B_2 to the left, and there is but one solution.

If A is right or obtuse, and $a = b$ or $< b$, neither B_1 nor B_2 can fall to the right of A, and the triangle is impossible.

All of these tests may be applied mentally to the data, except the test of the relative magnitudes of $b \sin A$ and a. The computation itself determines this test most satisfactorily; for, if $b \sin A$ is less than, equal to, or greater than a, $\sin B$, which equals $\frac{b \sin A}{a}$, will be respectively less than, equal to, or greater than unity; and consequently $\log \sin B$ will be negative, 0, or positive.

In solving any example under Case III., supposing the given parts to be A, a, and b, we should first examine whether A is acute, right, or obtuse. If either of the two latter, the triangle is impossible unless $a > b$, when there is one solution. If A is acute, and $a = b$ or $> b$, there is but one solution. If A is acute, and $a < b$, proceed with the computation until $\log \sin B$ has been found; if this is $-$, 0, or $+$, the example has two solutions, one, or none, respectively.

The value of B is never to be taken as obtuse unless there are two solutions, in which case both the acute and obtuse values are to be retained.

161. We will further illustrate the subject by a few examples.

1. Given $b = 3.39$, $c = 7.42$, $C = 105° 13'$; find B.

We have here $$\sin B = \frac{b \sin C}{c} \qquad \text{(Art. 144)}$$

or, $$\log \sin B = \log b + \operatorname{colog} c + \log \sin C$$

$$\begin{aligned} \log b &= 0.530200 \\ \operatorname{colog} c &= 9.129596 \\ \log \sin C &= \underline{9.984500} \\ \log \sin B &= 9.644296 \\ \therefore B &= 26° \, 9' \, 30.5'' \end{aligned}$$

there being but one solution, since $c > b$.

2. Given $c = .5$, $a = .227643$, $A = 27° \, 5'$; find C.

We have here $$\sin C = \frac{c \sin A}{a}$$

or, $$\log \sin C = \log c + \operatorname{colog} a + \log \sin A$$

$$\begin{aligned} \log c &= 9.698970 \\ \operatorname{colog} a &= 0.642746 \\ \log\sin A &= \underline{9.658284} \\ \log\sin C &= 0 \\ \therefore \sin C &= 1 \text{ (Art. 92), or } C = 90^\circ. \end{aligned}$$

Here there is but one solution; that is, a right triangle.

3. Given $a = .083$, $b = .0715$, $B = 61^\circ 47'$; find A.

We have here $$\sin A = \frac{a \sin B}{b}$$

or, $$\log\sin A = \log a + \operatorname{colog} b + \log\sin B$$

$$\begin{aligned} \log a &= 8.919078 \\ \operatorname{colog} b &= 1.145694 \\ \log\sin B &= \underline{9.945058} \\ \log\sin A &= 0.009830 \end{aligned}$$

Since this is positive, $\sin A > 1$, and the triangle is impossible.

162. CASE IV. *Given the three sides.*

The angles might be calculated by the formulæ of Art. 147; but as these are not adapted to logarithmic computation, it is preferable to use the formulæ of Art. 148. It is better to compute all of the angles by trigonometrical methods, for we then have a check on the work, since their sum must be 180°. If all the angles are to be calculated, the tangent formulæ will be found the most convenient, as only four different logarithms are required. If but one angle is to be found, the cosine formulæ are the best, since they involve the least work. The triangle is possible for any values of the data, provided no side is greater than the sum of the other two.

If all the angles are required, and the tangent formulæ are used, it will be found convenient to change their forms as follows:

By Art. 148,

$$\tan\tfrac{1}{2}A=\sqrt{\frac{(s-a)(s-b)(s-c)}{s(s-a)^2}}$$

$$=\frac{1}{s-a}\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$$

$$=\frac{r}{s-a}$$

where $$r=\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$$

Similarly, $\tan\frac{1}{2}B=\dfrac{r}{s-b}$, and $\tan\frac{1}{2}C=\dfrac{r}{s-c}$

Note. By Ex. 20, Art. 155, r is the radius of the inscribed circle.

1. Given $a=2.5$, $b=2.79$, $c=2.33$; find the angles.

Here $2s=a+b+c=7.62$

Hence, $s=3.81$, $s-a=1.31$, $s-b=1.02$, $s-c=1.48$

$$\log r=\tfrac{1}{2}[\log(s-a)+\log(s-b)+\log(s-c)+\operatorname{colog}s]$$

Also,
$$\begin{aligned}\log\tan\tfrac{1}{2}A&=\log r-\log(s-a)\\ \log\tan\tfrac{1}{2}B&=\log r-\log(s-b)\\ \log\tan\tfrac{1}{2}C&=\log r-\log(s-c)\end{aligned}$$

$$\begin{aligned}\log(s-a)&=0.117271\\ \log(s-b)&=0.008600\\ \log(s-c)&=0.170262\\ \operatorname{colog}s&=\underline{9.419075-10}\\ 2&\overline{)9.715208-10}\\ \therefore\ \log r&=9.857604-10\end{aligned}$$

$$\begin{aligned}\log r&=9.857604-10\\ \log(s-b)&=\underline{0.008600}\\ \log\tan\tfrac{1}{2}B&=9.849004-10\\ \tfrac{1}{2}B&=35^\circ\,14'\,4''\\ \therefore\ B&=70^\circ\,28'\,8''\end{aligned}$$

$$\begin{aligned}\log r&=9.857604-10\\ \log(s-a)&=\underline{0.117271}\\ \log\tan\tfrac{1}{2}A&=9.740333-10\\ \tfrac{1}{2}A&=28^\circ\,48'\,32.9''\\ \therefore\ A&=57^\circ\,37'\,5.8''\end{aligned}$$

$$\begin{aligned}\log r&=9.857604-10\\ \log(s-c)&=\underline{0.170262}\\ \log\tan\tfrac{1}{2}C&=9.687342-10\\ \tfrac{1}{2}C&=25^\circ\,57'\,23''\\ \therefore\ C&=51^\circ\,54'\,46''\end{aligned}$$

Check, $A+B+C=179^\circ\,59'\,59.8''$.

2. Given $a = 7$, $b = 11$, $c = 9.6$; find B.

By Art 148, $\cos\frac{1}{2}B = \sqrt{\dfrac{s(s-b)}{ca}}$

or, $\log\cos\frac{1}{2}B = \frac{1}{2}[\log s + \log(s-b) + \operatorname{colog} c + \operatorname{colog} a]$

We have $2s = a + b + c = 27.6$, or $s = 13.8$;

thus, $s - b = 2.8$

$$\begin{aligned} \log s &= 1.139879 \\ \log(s-b) &= 0.447158 \\ \operatorname{colog} c &= 9.017729 \\ \operatorname{colog} a &= 9.154902 \\ & 2\overline{)9.759668} \\ \log\cos\tfrac{1}{2}B &= 9.879834 \\ \tfrac{1}{2}B &= 40^\circ\,41'\,11.4'' \\ \therefore B &= 81^\circ\,22'\,22.8'' \end{aligned}$$

TO CALCULATE THE AREA OF A TRIANGLE.

163. *Example.* Given $a = 18.063$, $B = 35^\circ\,0'\,13''$, $A = 96^\circ\,30'\,15''$; find K.

By Art. 150,

$$2K = \frac{a^2 \sin B \sin C}{\sin A} = a^2 \sin B \sin C \operatorname{cosec} A$$

or, $\log(2K) = 2\log a + \log\sin B + \log\sin C + \log\operatorname{cosec} A$

From the data,

$$\begin{aligned} C = 180^\circ - (A + B) &= 180^\circ - 131^\circ\,30'\,28'' \\ &= 48^\circ\,29'\,32'' \end{aligned}$$

$$\begin{aligned} \log a = 1.256790 \qquad \therefore 2\log a &= 2.513580 \\ \log\sin B &= 9.758630 \\ \log\sin C &= 9.874404 \\ \log\operatorname{cosec} A &= 0.002805 \\ \hline \log(2K) &= 2.149419 \\ 2K &= 141.065 \\ \therefore K &= 70.5325 \end{aligned}$$

EXAMPLES.

164. Solve the following triangles:

41 1. Given $a = 10$, $A = 38°$, $B = 116° 7'$. 14.5843 7.09059

2. Given $a = 1.1$, $b = 1.5$, $c = 2.4$.

42 3. Given $a = 15$, $c = 27$, $B = 46°$. 33° 3′ 19.7″ 100° 56′ 40.3″ 19.282

44 4. Given $a = 1.71$, $c = 1.07$, $A = 31° 53'$.

14 5. Given $a = .17$, $c = .1864$, $A = 63° 40'$.

42 6. Given $a = .23017$, $c = .3567$, $B = 61° 58'$. 39° 15′ 56.2″ ,320999

41 7. Given $a = .02901$, $B = 89° 40'$, $C = 33° 15'$..0345573 .0189474

8. Given $a = .23$, $b = .26$, $c = .198$.

41 9. Given $a = 5.42$, $B = 98° 22'$, $C = 41° 1'$.8.23709, 5.46392

44 10. Given $a = 50.28$, $b = 66.82$, $A = 32° 49'$.

42 11. Given $a = 4.86$, $b = 3.47$, $C = 51° 36'$. 83° 14′ 37.2″, 3.83539

12. Given $a = 79.3$, $b = 94.2$, $c = 66.9$.

41 13. Given $a = 1.61$, $A = 35° 15'$, $C = 123° 39'$. 1.00425 2.32217

45 14. Given $b = 6.84$, $c = 5.161$, $C = 44° 3'$.

45 15. Given $b = .70968$, $c = .5112$, $B = 35° 11' 16''$.

41 16. Given $c = 400$, $A = 65° 31' 48''$, $B = 60°$. 447.366, 425.664

42 17. Given $a = .3$, $b = .363$, $C = 54° 56'$. 52° 10′ 22.3″ ,.310884

42 18. Given $b = 314.286$, $A = 67° 21' 42''$, $C = 54° 47' 12''$. 342.602
303.276

19. Given $a = 3.21$, $b = 3.61$, $c = 4.02$.

45 20. Given $a = 10$, $b = 5.007556$, $B = 30° 3'$.

54 21. Given $a = .03563$, $b = .11921$, $C = 6° 16' 10''$.

42 22. Given $c = 78.60124$, $A = 104° 31' 9''$, $B = 32° 2' 52''$. 110.677
60.6656

44 23. Given $b = 74.00206$, $c = 114.3851$, $A = 32° 25' 46''$.

45 24. Given $a = 106.85268$, $c = 166.2131$, $A = 40° 0' 21''$.

45 25. Given $b = 3068.7$, $c = 1223.29$, $C = 55° 11' 18''$.

26. Given $a = 641$, $b = 529$, $c = 702$.

45 **27.** Given $b = 214.5637$, $c = 284.7906$, $B = 44° 19' 53''$.

Find the area in each of the following:

5112.25 40 **28.** Given $b = 116.082$, $c = 100$, $A = 118° 15' 41''$.

2624.07

3506.815 **29.** Given $a = 79.5$, $b = 94.4$, $c = 67.1$.

30. Given $b = 100$, $C = 40° 5'$, $A = 76° 38' 13''$.

135.8545 **31.** Given $a = 31.235$, $B = 13° 57' 2''$, $A = 53° 11' 18''$.

21595.40 **32.** Given $a = 231$, $b = 197$, $c = 252$.

12.7266 1 **33.** Given $c = 9.08072$, $a = 3.2281$, $B = 60° 15' 46''$.

46.17725 40 **34.** Given $c = 8.024761$, $B = 100° 5' 23''$, $C = 31° 6' 12''$.

.00181672 41 **35.** Given $a = .1016837$, $b = .1822451$, $C = 11° 18' 26''$.

.117655 41 **36.** Given $a = .582$, $b = .601$, $c = .427$.

37. From the top of a lighthouse, 200 feet above the sea level, the angles of depression of two boats in line with the lighthouse are observed to be 14° and 32°, respectively. What is the distance between the boats?

38. From a point in the same horizontal plane with the base of a tower, the angle of elevation of its top is 52° 39′; and from a point 100 feet further away it is 35° 16′. Required the height of the tower.

39. The sides of a field $ABCD$ are $AB = 155$ feet, $BC = 236$ feet, $CD = 252$ feet, and $DA = 105$ feet, and the length of a line from A to C is 311 feet. Find the area of the field.

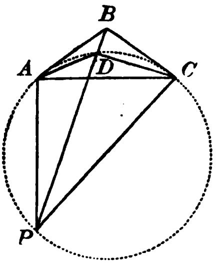
40. Wishing to obtain the distance between two buoys, A and B, I measure a base line CD on the shore, 150 feet in length; at the point C the angles ACD and BCD are measured and found to be 95° and 70°, respectively; and at the point D, the angles BDC and ADC are 83° and 30°, respectively. What is the distance between the buoys?

41. In a field $ABCD$, the sides AB, BC, and DA are 37, 63, and 20 rods, respectively; and the diagonals AC and BD are 75 and 42 rods, respectively. Find the area.

42. From the top of a bluff the angles of depression of two posts in the plain below, in line with the observer, and 1000 feet apart, are found to be 27° 40′ and 9° 33′, respectively. What is the height of the bluff above the plain?

43. From a boat, three church steeples on the shore, A, B, and C, are observed to bear due north and 21° and 44° east of north, respectively. If the distances AB, BC, and CA are 3000, 3500, and 5870 feet, respectively, find the distance of the boat from each steeple.

Note. If P denote the position of the boat, describe a circumference passing through A, C, and P, and meeting PB at D. By Geometry, the angles DAC and DCA are equal to the angles DPC and DPA, respectively, hence, in the triangle ADC two angles and a side are known, and the side AD may be determined. Again, in the triangle ABC, the three sides are known, and therefore the angle BAC may be found; and hence, also, the angle BAD, which is the difference of the angles BAC and DAC. Then, in the triangle BAD, we know two sides and their included angle, whence we may calculate the angle ABD; and finally, in the triangle ABP, a side and two angles being known, the sides PA and PB can be determined. After PA and PB are found, the calculation of PC is readily effected.



XI. SOLUTION OF NUMERICAL EQUATIONS OF THE THIRD DEGREE.

165. We know, by Algebra, that a cubic equation can always be transformed into another in which the terms containing the square of the unknown quantity are wanting.

Thus, if the equation is $x^3+px^2+qx+r=0$,

putting $x=y-\frac{p}{3}$, we have

$$y^3-py^2+\frac{p^2y}{3}-\frac{p^3}{27}+py^2-\frac{2p^2y}{3}+\frac{p^3}{9}+qy-\frac{pq}{3}+r=0$$

or, $$y^3+y\left(q-\frac{p^2}{3}\right)+r-\frac{pq}{3}+\frac{2p^3}{27}=0$$

which is in the required form.

166. Cardan's Method enables us to solve any cubic equation in the form $x^3+ax+b=0$, except in the case where a is negative and $-\frac{a^3}{27}>\frac{b^2}{4}$.

In this case the three roots are real and unequal, and Cardan's Method involves the cube root of a binomial surd, which cannot be obtained by Algebra. It is possible, however, to find the roots by Trigonometrical methods.

167. *To solve the equation*

$$x^3-ax-b=0$$

when a is positive, b positive or negative, and $\frac{a^3}{27}>\frac{b^2}{4}$.

Putting $x=my$, $$m^3y^3-amy-b=0$$

or, $$y^3-\frac{ay}{m^2}-\frac{b}{m^3}=0 \qquad (\text{A})$$

But, $$4\cos^3A-3\cos A-\cos 3A=0 \qquad (\text{Art. } 71)$$

That is, $$\cos^3A-\tfrac{3}{4}\cos A-\tfrac{1}{4}\cos 3A=0$$

This is identical with (A) provided

$$y=\cos A,\quad \frac{a}{m^2}=\frac{3}{4},\quad \text{and}\quad \frac{b}{m^3}=\frac{1}{4}\cos 3A \qquad (\text{B})$$

From (B), $$m^2=\frac{4a}{3}\quad \text{or}\quad m=\pm\sqrt{\frac{4a}{3}} \qquad (\text{C})$$

Substituting the positive value of m in the third of (B),

$$\cos 3A = \frac{4b}{m^3} = 4b\left(\sqrt{\frac{3}{4a}}\right)^3 = \frac{b}{2}\sqrt{\frac{27}{a^3}} \qquad \text{(D)}$$

Since, by hypothesis, $\frac{a^3}{27} > \frac{b^2}{4}$, we have $\frac{b^2}{4} \times \frac{27}{a^3} < 1$; and taking the square roots of both members of the inequality, $\frac{b}{2}\sqrt{\frac{27}{a^3}} < 1$. We thus see that the value of $3A$ in (D) is possible, since its cosine is less than unity.

Let z be the least positive angle whose cosine is equal to $\frac{b}{2}\sqrt{\frac{27}{a^3}}$; then one value of $3A$ is z, and all its values are given by the expression $2n\pi \pm z$ (Art. 52), where $n = 0$ or any positive or negative integer.

Then, by (B), $y = \cos A = \cos \frac{1}{3}\{2n\pi \pm z\}$.

But the latter expression has only the three *different* values obtained by giving to n the values 0 and ± 1; for, if n has any other integral value, say $3q \pm n'$, where n' is 0 or ± 1, and q is any positive or negative integer,

$$\begin{aligned} \cos \tfrac{1}{3}\{2n\pi \pm z\} &= \cos \tfrac{1}{3}\{2(3q \pm n')\pi \pm z\} \\ &= \cos\{2q\pi \pm \tfrac{1}{3}(2n'\pi \pm z)\} \\ &= \cos \tfrac{1}{3}\{2n'\pi \pm z\} \qquad \text{(Art. 49)} \end{aligned}$$

all values of which are included among the results obtained by giving to n' the values 0 and ± 1. Thus, only three different values can be found for y; that is (Art. 40),

$$\cos \tfrac{1}{3}z,\ \cos \tfrac{1}{3}(2\pi - z),\ \text{and}\ \cos \tfrac{1}{3}(2\pi + z)$$

But $x = my$, and $m = \sqrt{\frac{4a}{3}}$; hence the values of x are

$$\sqrt{\frac{4a}{3}}\cos\frac{z}{3},\ \sqrt{\frac{4a}{3}}\cos\left(\frac{2\pi}{3} - \frac{z}{3}\right),\ \text{and}\ \sqrt{\frac{4a}{3}}\cos\left(\frac{2\pi}{3} + \frac{z}{3}\right),$$

where z is given by the equation $\cos z = \frac{b}{2}\sqrt{\frac{27}{a^3}}$.

If the negative value of m is taken in equation (**C**), we shall find for y the values

$$-\cos\tfrac{1}{3}z,\quad -\cos\tfrac{1}{3}(2\pi-z),\ \text{and}\ -\cos\tfrac{1}{3}(2\pi+z),$$

where z denotes, as before, the least positive angle whose cosine equals $\frac{b}{2}\sqrt{\frac{27}{a^3}}$. But since m is now $-\sqrt{\frac{4a}{3}}$, we obtain the same values for x as before.

EXAMPLES.

1. Solve the equation $x^3-4x+2=0$.

Here $a=4$, $b=-2$. Then, $\cos z=-\sqrt{\frac{27}{64}}=-\frac{3}{8}\sqrt{3}$.

Or (Art. 43), $\cos(\pi-z)=\frac{3}{8}\sqrt{3}$

Thus, $\log\cos(\pi-z)=\log 3+\frac{1}{2}\log 3+\operatorname{colog} 8$

$$\begin{aligned}
\log 3 &= .477121\\
\tfrac{1}{2}\log 3 &= .238560\\
\operatorname{colog} 8 &= \underline{9.096910-10}\\
\log\cos(\pi-z) &= 9.812591-10\\
\therefore\ \pi-z &= 49^\circ\ 29'\ 41.1''\\
\therefore\ z &= 130^\circ\ 30'\ 18.9''
\end{aligned}$$

Also, $\sqrt{\frac{4a}{3}}=\sqrt{\frac{16}{3}}=\frac{4}{\sqrt{3}}$

Hence, the three values of x are

$$\begin{aligned}
\frac{4}{\sqrt{3}}\cos\frac{z}{3} &= \frac{4}{\sqrt{3}}\cos 43^\circ\ 30'\ 6.3''\\
\frac{4}{\sqrt{3}}\cos\left(\frac{2\pi}{3}-\frac{z}{3}\right) &= \frac{4}{\sqrt{3}}\cos 76^\circ\ 29'\ 53.7''\\
\frac{4}{\sqrt{3}}\cos\left(\frac{2\pi}{3}+\frac{z}{3}\right) &= \frac{4}{\sqrt{3}}\cos 163^\circ\ 30'\ 6.3''\\
&= -\frac{4}{\sqrt{3}}\sin 73^\circ\ 30'\ 6.3''
\end{aligned}$$

Now,
$$\log\frac{4}{\sqrt{3}} = \log 4 - \tfrac{1}{2}\log 3$$
$$= .602060 - .238560$$
$$= .363500 \qquad (1)$$
$$\log\cos 43^\circ\, 30'\ \ 6.3'' = 9.860549 - 10 \qquad (2)$$
$$\log\cos 76^\circ\, 29'\, 53.7'' = 9.368241 - 10 \qquad (3)$$
$$\log\sin\ 73^\circ\, 30'\ \ 6.3'' = 9.981741 - 10 \qquad (4)$$

Adding (2), (3), and (4) successively to (1), we have for the logarithms of the values of x

0.224049, 9.731741–10, and 0.345241.

The numbers corresponding to these are

1.67513, .539189, and 2.21432.

Hence, $x = 1.67513$, .539189, or -2.21432. *Ans.*

Solve the following equations:

2. $x^3 - 4x - 1 = 0.$

3. $x^3 - 6x - 3 = 0.$

4. $x^3 + 6x^2 - x - 1 = 0.$

5. $x^3 - 3x^2 - 2x + 1 = 0.$

XII. DE MOIVRE'S THEOREM,

WITH APPLICATIONS.

168. DE MOIVRE'S THEOREM. *For any value of n, positive or negative, integral or fractional,*

$$(\cos x + \sqrt{-1}\sin x)^n = \cos nx + \sqrt{-1}\sin nx \qquad \mathbf{(114)}$$

CASE I. *When n is a positive integer.*

We have

$$(\cos x + \sqrt{-1}\ \sin x)^2$$
$$= \cos^2 x - \sin^2 x + 2\sqrt{-1}\ \sin x \cos x$$
$$= \cos 2x + \sqrt{-1}\ \sin 2x, \text{ by Art. 67.} \qquad (\text{A})$$

Again,

$$\begin{aligned}
&(\cos x + \sqrt{-1}\sin x)^3\\
&\quad = (\cos x + \sqrt{-1}\sin x)^2(\cos x + \sqrt{-1}\sin x)\\
&\quad = (\cos 2x + \sqrt{-1}\sin 2x)(\cos x + \sqrt{-1}\sin x), \text{ by } (\mathrm{A})\\
&\quad = \cos 2x \cos x - \sin 2x \sin x\\
&\qquad\qquad + \sqrt{-1}(\sin 2x \cos x + \cos 2x \sin x)\\
&\quad = \cos(2x + x) + \sqrt{-1}\sin(2x + x) \qquad (\text{Art. } 57)\\
&\quad = \cos 3x + \sqrt{-1}\sin 3x
\end{aligned}$$

The theorem is thus seen to hold when $n = 2$ or 3.

The general proof for any positive integral exponent may be obtained as follows:

Suppose the theorem to hold when $n = r$, r being any positive integer. Then,

$$(\cos x + \sqrt{-1}\sin x)^r = \cos rx + \sqrt{-1}\sin rx$$

Multiplying both members of this equation by

$$\cos x + \sqrt{-1}\sin x$$

we have

$$\begin{aligned}
&(\cos x + \sqrt{-1}\sin x)^{r+1}\\
&\quad = (\cos rx + \sqrt{-1}\sin rx)(\cos x + \sqrt{-1}\sin x)\\
&\quad = \cos rx \cos x - \sin rx \sin x\\
&\qquad\qquad + \sqrt{-1}(\sin rx \cos x + \cos rx \sin x)\\
&\quad = \cos(rx + x) + \sqrt{-1}\sin(rx + x)\\
&\quad = \cos(r + 1)x + \sqrt{-1}\sin(r + 1)x
\end{aligned}$$

which result is seen to accord with the theorem.

That is, if the theorem holds for any positive integral value of n, it holds for a value greater by unity. But it has already been proved to hold when $n = 3$, hence it holds when $n = 4$; hence, also, when $n = 5$; and so on. Thus, the theorem is true when n is any positive integer.

Case II. *When n is any negative integer.*

Let $n = -m$, where m is a positive integer. Then

$$(\cos x+\sqrt{-1}\,\sin x)^{n}=(\cos x+\sqrt{-1}\,\sin x)^{-m}$$

$$=\frac{1}{(\cos x+\sqrt{-1}\,\sin x)^{m}}$$

$$=\frac{1}{\cos mx+\sqrt{-1}\,\sin mx}\quad\text{by Case I.}$$

But, by Art. 19,

$$1=\cos^2 mx+\sin^2 mx=\cos^2 mx-(\sqrt{-1}\,\sin mx)^2$$

Hence,

$$(\cos x+\sqrt{-1}\,\sin x)^{-m}=\frac{\cos^2 mx-(\sqrt{-1}\,\sin mx)^2}{\cos mx+\sqrt{-1}\,\sin mx}$$

$$=\cos mx-\sqrt{-1}\,\sin mx$$

$$=\cos(-mx)+\sqrt{-1}\,\sin(-mx)$$

(Art. 40)

which was to be proved.

Case III. *When n is a fraction.*

Let $n=\frac{p}{q}$, where p and q are any integers, positive or negative.

Now, by Cases I. and II.,

$$(\cos x+\sqrt{-1}\,\sin x)^{p}=\cos px+\sqrt{-1}\,\sin px$$

and, $\left(\cos\frac{p}{q}x+\sqrt{-1}\,\sin\frac{p}{q}x\right)^{q}=\cos px+\sqrt{-1}\,\sin px$

Hence, $(\cos x+\sqrt{-1}\,\sin x)^{p}=\left(\cos\frac{p}{q}x+\sqrt{-1}\,\sin\frac{p}{q}x\right)^{q}$

Taking the qth root of both members,

$$(\cos x+\sqrt{-1}\,\sin x)^{\frac{p}{q}}=\cos\frac{p}{q}x+\sqrt{-1}\,\sin\frac{p}{q}x$$

which was to be proved.

169. Putting $-x$ for x in De Moivre's Theorem, we have

$$\{\cos(-x)+\sqrt{-1}\sin(-x)\}^n=\cos(-nx)+\sqrt{-1}\sin(-nx)$$

or, by Art. 40,

$$(\cos x-\sqrt{-1}\sin x)^n=\cos nx-\sqrt{-1}\sin nx \qquad \textbf{(115)}$$

which is another form of the theorem. The two forms may be united in one, as follows:

$$(\cos x\pm\sqrt{-1}\sin x)^n=\cos nx\pm\sqrt{-1}\sin nx \qquad \textbf{(116)}$$

where the upper signs or the lower signs must be taken together.

170. It is to be observed that, when n is a fraction equal to $\frac{p}{q}$, $\cos\frac{p}{q}x+\sqrt{-1}\sin\frac{p}{q}x$ is *only one of the q different values* of the expression $(\cos x+\sqrt{-1}\sin x)^{\frac{p}{q}}$.

For, if k is any positive integer,

$$\begin{aligned}&\left\{\cos\frac{p(x+2k\pi)}{q}+\sqrt{-1}\sin\frac{p(x+2k\pi)}{q}\right\}^q\\ &=\cos p(x+2k\pi)+\sqrt{-1}\sin p(x+2k\pi)\\ &=\cos(2pk\pi+px)+\sqrt{-1}\sin(2pk\pi+px)\\ &=\cos px+\sqrt{-1}\sin px \qquad \text{(Art. 49)}\end{aligned}$$

Thus, by the proof of Case III., Art. 168, we see that

$$\begin{aligned}&(\cos x+\sqrt{-1}\sin x)^{\frac{p}{q}}\\ &\qquad=\cos\frac{p(x+2k\pi)}{q}+\sqrt{-1}\sin\frac{p(x+2k\pi)}{q}\end{aligned}$$

The expression in the right hand member has only the q *different* values obtained by giving to k the values $0, 1, 2, \cdots, q-1$. For, if k has any other value, say $mq+k'$, where k' is any term of the series $0, 1, 2, \cdots, q-1$, and m is any positive or negative integer,

$$\cos\frac{p(x+2k\pi)}{q}+\sqrt{-1}\sin\frac{p(x+2k\pi)}{q}$$

$$=\cos\frac{p\{x+2(mq+k')\pi\}}{q}+\sqrt{-1}\sin\frac{p\{x+2(mq+k')\pi\}}{q}$$

$$=\cos\left\{2mp\pi+\frac{p(x+2k'\pi)}{q}\right\}$$

$$+\sqrt{-1}\sin\left\{2mp\pi+\frac{p(x+2k'\pi)}{q}\right\}$$

$$=\cos\frac{p(x+2k'\pi)}{q}+\sqrt{-1}\sin\frac{p(x+2k'\pi)}{q}$$

which is among the expressions obtained by giving to k the values 0, 1, 2, ⋯, $q-1$. Thus only q different values can be found for the expression. This agrees with the well-known principle of Algebra that an expression has q qth roots.

To illustrate, the three values of $(\cos x+\sqrt{-1}\sin x)^{\frac{2}{3}}$ are

$$\cos\tfrac{2}{3}x+\sqrt{-1}\sin\tfrac{2}{3}x$$

$$\cos\frac{2(x+2\pi)}{3}+\sqrt{-1}\sin\frac{2(x+2\pi)}{3}$$

and $$\cos\frac{2(x+4\pi)}{3}+\sqrt{-1}\sin\frac{2(x+4\pi)}{3}$$

That is, $\cos\tfrac{2}{3}x+\sqrt{-1}\sin\tfrac{2}{3}x$

$$\cos\left(\frac{4\pi}{3}+\frac{2x}{3}\right)+\sqrt{-1}\sin\left(\frac{4\pi}{3}+\frac{2x}{3}\right)$$

and $$\cos\left(\frac{8\pi}{3}+\frac{2x}{3}\right)+\sqrt{-1}\sin\left(\frac{8\pi}{3}+\frac{2x}{3}\right)$$

The last expression may also be written by Art. 49;

$$\cos\left(\frac{2\pi}{3}+\frac{2x}{3}\right)+\sqrt{-1}\sin\left(\frac{2\pi}{3}+\frac{2x}{3}\right)$$

EXPANSIONS OF THE SINE AND COSINE.

171. By De Moivre's Theorem,

$$\cos ny+\sqrt{-1}\,\sin ny=(\cos y+\sqrt{-1}\,\sin y)^n \qquad (\text{A})$$

Expanding the second member by the Binomial Theorem, we have

$$\cos^n y+n\sqrt{-1}\,\cos^{n-1}y\,\sin y-\frac{n\,(n-1)}{\underline{|2}}\cos^{n-2}y\,\sin^2 y$$

$$-\frac{n\,(n-1)\,(n-2)}{\underline{|3}}\sqrt{-1}\,\cos^{n-3}y\,\sin^3 y+\cdots$$

$$=\cos^n y-\frac{n\,(n-1)}{\underline{|2}}\cos^{n-2}y\,\sin^2 y+\cdots$$

$$+\sqrt{-1}\left\{n\cos^{n-1}y\,\sin y-\frac{n\,(n-1)\,(n-2)}{\underline{|3}}\cos^{n-3}y\,\sin^3 y+\cdots\right\}$$

By Algebra, the real and imaginary portions of the two members of equation (A) must be equal, and we have

$$\cos ny=\cos^n y-\frac{n\,(n-1)}{\underline{|2}}\cos^{n-2}y\,\sin^2 y+\cdots$$

$$\sin ny=n\cos^{n-1}y\sin y-\frac{n\,(n-1)\,(n-2)}{\underline{|3}}\cos^{n-3}y\,\sin^3 y+\cdots$$

That is,

$$\cos ny=\cos^n y\left\{1-\frac{n\,(n-1)}{\underline{|2}}\tan^2 y+\frac{n\,(n-1)\,(n-2)\,(n-3)}{\underline{|4}}\tan^4 y-\cdots\right\}$$

$$\sin ny=\cos^n y\left\{n\tan y-\frac{n\,(n-1)\,(n-2)}{\underline{|3}}\tan^3 y+\cdots\right\}$$

Putting $ny=x$, and consequently $y=\frac{x}{n}$, we have

$$\cos x = \left(\cos\frac{x}{n}\right)^n \left\{ 1 - \frac{n(n-1)}{\underline{|2}}\tan^2\frac{x}{n} + \frac{n(n-1)(n-2)(n-3)}{\underline{|4}}\tan^4\frac{x}{n} - \cdots \right\}$$

$$\sin x = \left(\cos\frac{x}{n}\right)^n \left\{ n\tan\frac{x}{n} - \frac{n(n-)1(n-2)}{\underline{|3}}\tan^3\frac{x}{n} + \cdots \right\}$$

These may be written,

$$\cos x = \left(\cos\frac{x}{n}\right)^n \left\{ 1 - \frac{x^2}{\underline{|2}}\left(1-\frac{1}{n}\right)\frac{\tan^2\frac{x}{n}}{\frac{x^2}{n^2}} + \frac{x^4}{\underline{|4}}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(1-\frac{3}{n}\right)\frac{\tan^4\frac{x}{n}}{\frac{x^4}{n^4}} - \cdots \right\}$$

$$\sin x = \left(\cos\frac{x}{n}\right)^n \left\{ x\frac{\tan\frac{x}{n}}{\frac{x}{n}} - \frac{x^3}{\underline{|3}}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\frac{\tan^3\frac{x}{n}}{\frac{x^3}{n^3}} + \cdots \right\}$$

That is,

$$\cos x = \left(\cos\frac{x}{n}\right)^n \left\{ 1 - \frac{x^2}{\underline{|2}}\left(1-\frac{1}{n}\right)\left(\frac{\tan\frac{x}{n}}{\frac{x}{n}}\right)^2 + \frac{x^4}{\underline{|4}}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(1-\frac{3}{n}\right)\left(\frac{\tan\frac{x}{n}}{\frac{x}{n}}\right)^4 - \cdots \right\}$$

$$\sin x = \left(\cos\frac{x}{n}\right)^n \left\{ x\frac{\tan\frac{x}{n}}{\frac{x}{n}} - \frac{x^3}{\underline{|3}}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(\frac{\tan\frac{x}{n}}{\frac{x}{n}}\right)^3 + \cdots \right\}$$

Let, now, n be indefinitely increased, and consequently $\frac{x}{n}$ indefinitely diminished. By Art. 80, $\dfrac{\tan\frac{x}{n}}{\frac{x}{n}}$ approaches the limit 1, and by Art. 81, $\left(\cos\frac{x}{n}\right)^n$ approaches the same limit; also, each of the quantities $1-\frac{1}{n}$, $1-\frac{2}{n}$, etc., approaches the limit 1. Hence, at the limit, we have

$$\cos x = 1 - \frac{x^2}{\underline{|2}} + \frac{x^4}{\underline{|4}} - \cdots \tag{117}$$

$$\sin x = x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} - \cdots \tag{118}$$

Note. It is evident that, in the series for $\sin x$ and $\cos x$ just found, x must be expressed in *circular* measure.

EXPONENTIAL VALUES OF THE SINE AND COSINE.

172. By Art. 110, putting $x\sqrt{-1}$ for x,

$$e^{x\sqrt{-1}} = 1 + x\sqrt{-1} - \frac{x^2}{\underline{|2}} - \frac{x^3\sqrt{-1}}{\underline{|3}} + \frac{x^4}{\underline{|4}} + \frac{x^5\sqrt{-1}}{\underline{|5}} - \cdots$$

$$= 1 - \frac{x^2}{\underline{|2}} + \frac{x^4}{\underline{|4}} - \cdots + \sqrt{-1}\left\{x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} - \cdots\right\}$$

$$= \cos x + \sqrt{-1}\sin x. \quad \text{(Art. 171.)} \tag{119}$$

Putting in this result $-x$ in place of x,

$$e^{-x\sqrt{-1}} = \cos(-x) + \sqrt{-1}\sin(-x)$$
$$= \cos x - \sqrt{-1}\sin x \tag{120}$$

173. Adding equations (**119**) and (**120**), we have

$$2\cos x = e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}$$

Whence, $$\cos x = \frac{e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}}{2} \tag{121}$$

Subtracting (**120**) from (**119**), we have

$$2\sqrt{-1}\sin x = e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}$$

Whence, $$\sin x = \frac{e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}}{2\sqrt{-1}} \tag{122}$$

These results are called the *exponential*, or *hyperbolic*, values of the sine and cosine; they were first obtained by Euler.

Dividing (**122**) by (**121**), we have

$$\tan x = \frac{\dfrac{e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}}{2\sqrt{-1}}}{\dfrac{e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}}{2}} = \frac{e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}}{\sqrt{-1}\,(e^{x\sqrt{-1}} + e^{-x\sqrt{-1}})} \qquad (\mathbf{123})$$

SERIES FOR CALCULATING THE VALUE OF π.

174. Gregory's Series. *To expand x in a series of powers of* $\tan x$.

By Art. 173,

$$\frac{1+\sqrt{-1}\tan x}{1-\sqrt{-1}\tan x} = \frac{1+\dfrac{e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}}{e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}}}{1-\dfrac{e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}}{e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}}} = \frac{2e^{x\sqrt{-1}}}{2e^{-x\sqrt{-1}}} = e^{2x\sqrt{-1}}$$

Whence, by logarithms,

$$\log_e (e^{2x\sqrt{-1}}) = \log_e \frac{1+\sqrt{-1}\tan x}{1-\sqrt{-1}\tan x}$$

or, $2x\sqrt{-1} = \log_e (1+\sqrt{-1}\tan x) - \log_e (1-\sqrt{-1}\tan x)$

which, by Art. 113,

$$= \sqrt{-1}\tan x + \frac{\tan^2 x}{2} - \sqrt{-1}\,\frac{\tan^3 x}{3} - \cdots$$

$$+ \sqrt{-1}\tan x - \frac{\tan^2 x}{2} - \sqrt{-1}\,\frac{\tan^3 x}{3} + \cdots$$

$$= 2\sqrt{-1}\left\{\tan x - \frac{\tan^3 x}{3} + \frac{\tan^5 x}{5} - \cdots\right\}$$

That is, $x = \tan x - \dfrac{\tan^3 x}{3} + \dfrac{\tan^5 x}{5} - \cdots \qquad (\mathbf{124})$

175. Putting in Gregory's Series $\tan x = y$, whence $x = \tan^{-1} y$ (Art. 82), we have

$$\tan^{-1} y = y - \frac{y^3}{3} + \frac{y^5}{5} - \frac{y^7}{7} + \cdots \qquad (\mathbf{125})$$

In this, as in all similar series, the angle $\tan^{-1}y$ must be expressed in circular measure.

Put $y = 1$; then, since the angle whose tangent is 1 is 45°, or $\frac{\pi}{4}$ when expressed in circular measure, we have

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \tag{126}$$

This series might be used to calculate the approximate value of π, but it converges so slowly as to require the use of a great many terms to obtain even the ordinary approximation, 3.1416.

176. EULER'S SERIES.

By Ex. 2, Art. 82,

$$\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = \frac{\pi}{4}$$

Whence (Art. 175),

$$\frac{\pi}{4} = \frac{1}{2} - \frac{1}{3.2^3} + \frac{1}{5.2^5} - \cdots + \frac{1}{3} - \frac{1}{3.3^3} + \frac{1}{5.3^5} - \cdots \tag{127}$$

a series which converges far more rapidly than that of the preceding article.

177. MACHIN'S SERIES.

By Art. 67, $\tan 2x = \dfrac{2\tan x}{1 - \tan^2 x}$

Putting $\tan x = \frac{1}{5}$, we have

$$\tan 2x = \frac{\frac{2}{5}}{1 - \frac{1}{25}} = \frac{2}{5} \div \frac{24}{25} = \frac{5}{12}$$

and $$\tan 4x = \frac{2\tan 2x}{1 - \tan^2 2x} = \frac{\frac{5}{6}}{1 - \frac{25}{144}} = \frac{5}{6} \div \frac{119}{144} = \frac{120}{119}$$

We thus see that $4x$ is a little greater than $\frac{\pi}{4}$, since its tangent is a little greater than 1. Then, let

$$4x = \frac{\pi}{4} + y$$

whence, $\tan 4x = \tan\left(\frac{\pi}{4} + y\right)$

or (Art. 63), $\frac{120}{119} = \frac{1 + \tan y}{1 - \tan y}$

Clearing of fractions,

$$120 - 120\tan y = 119 + 119\tan y$$

whence, $\tan y = \frac{1}{239}$, or $y = \tan^{-1}\frac{1}{239}$

Also, since $\tan x = \frac{1}{5}$, we have $x = \tan^{-1}\frac{1}{5}$

But, from the work above, we have

$$\frac{\pi}{4} = 4x - y$$

Hence, by substitution,

$$\frac{\pi}{4} = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239} \qquad (128)$$

Therefore, by Art. 175,

$$\frac{\pi}{4} = 4\left\{\frac{1}{5} - \frac{1}{3.5^3} + \frac{1}{5.5^5} - \frac{1}{7.5^7} + \cdots\right\}$$
$$- \left\{\frac{1}{239} - \frac{1}{3.239^3} + \frac{1}{5.239^5} - \cdots\right\} \qquad (129)$$

178. The series obtained in the preceding article converges very rapidly indeed, and it only requires the first three terms of the first line, and the first term of the second line, to give the approximate value of π, correct to the fourth decimal place.

Thus, approximately,

$$\frac{\pi}{4}=4\left(\frac{1}{5}-\frac{1}{3.5^3}+\frac{1}{5.5^5}\right)-\frac{1}{239}$$

$$=4\,(.2-.0027+.0001)-.0042$$

$$=.7896-.0042=.7854$$

Hence, $$\pi=4\times .7854=3.1416.$$

179. Let $$\left.\begin{aligned}\tan a&=\frac{1}{70}\\ \tan b&=\frac{1}{99}\end{aligned}\right\}\text{ whence, }\left\{\begin{aligned}a&=\tan^{-1}\frac{1}{70}\\ b&=\tan^{-1}\frac{1}{99}\end{aligned}\right.$$

Then,

$$\tan(a-b)=\frac{\tan a-\tan b}{1+\tan a\tan b}$$

$$=\frac{\frac{1}{70}-\frac{1}{99}}{1+\frac{1}{6930}}=\frac{1}{239}$$

That is, $$a-b=\tan^{-1}\frac{1}{239}$$

Hence, $\tan^{-1}\frac{1}{239}=\tan^{-1}\frac{1}{70}-\tan^{-1}\frac{1}{99}$

Whence, by Art. 177,

$$\frac{\pi}{4}=4\tan^{-1}\frac{1}{5}-\tan^{-1}\frac{1}{70}+\tan^{-1}\frac{1}{99} \qquad \textbf{(130)}$$

It is more convenient in the calculation of the approximate value of π to use the series for $\tan^{-1}\frac{1}{70}$ and $\tan^{-1}\frac{1}{99}$ than to use that for $\tan^{-1}\frac{1}{239}$, as in Art. 178.

PART II.

SPHERICAL TRIGONOMETRY.

XIII. GEOMETRICAL DEFINITIONS AND PRINCIPLES.

180. We know by Geometry that if a triedral angle be formed with its vertex at the centre of a sphere, it intercepts on the surface a *spherical triangle.*

181. The triangle is bounded by three arcs of great circles called its *sides*, which are measured by the face angles of the triedral angle.

Thus, the sides, as well as the angles, of a spherical triangle are expressed in angular measure. If the length of any side in terms of some *linear* unit is desired, it may readily be found by finding the ratio of its arc to 360°, and multiplying the result by the length of the circumference of a great circle.

That is, if A be the number of degrees in the side, and R the radius of the sphere, the length of the side is

$$\frac{A}{360} \times 2\pi R.$$

182. The diedral angles of the triedral angle are called the *angles* of the spherical triangle; and, by Geometry, each is measured by the angle between two straight lines drawn, one in each of its faces, and perpendicular to its edge at the same point.

183. Spherical Trigonometry treats of the trigonometrical relations between the elements of a spherical triangle; or, what is the same thing, between the face and diedral angles of the triedral angle which intercepts it.

184. It is evident that the face and diedral angles are not altered in magnitude by varying the radius of the sphere; and consequently the relations between the sides and angles of a spherical triangle are entirely independent of the length of the radius.

185. We shall limit ourselves in this work to such triangles as are considered in Geometry, where each angle is less than two right angles, and each side is less than the semi-circumference of a great circle; that is, where each element is less than 180°.

But in some of the higher applications of the subject, especially in the general formulæ of Astronomy and Geodesy, it is sometimes convenient to consider what is called the *general spherical triangle*, where an element may have any value between 0° and 360°.

186. The proofs of the following elementary properties of spherical triangles may be found in any treatise on Solid Geometry:

(*a*). Any side of a spherical triangle is less than the sum of the other two.

(*b*). If two sides of a spherical triangle are equal, the angles opposite them are equal; and conversely.

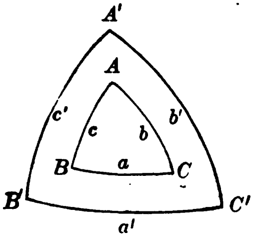
(*c*). If two sides of a spherical triangle are unequal, the angles opposite them are unequal, and the greater angle lies opposite the greater side; and conversely.

(*d*). The perpendicular from the vertex to the base of an isosceles spherical triangle bisects the base and the vertical angle, and divides it into two symmetrical right triangles.

(*e*). The sum of the sides of a spherical triangle is less ian the circumference of a great circle; that is, less than 30°.

(*f*). The sum of the angles of a spherical triangle is reater than two and less than six right angles; that is, reater than 180° and less than 540°.

(*g*). If $A'B'C'$ is the polar triangle of ABC, that is, if A, , and C are the poles of the arcs a', b', and c', respectively, ien, conversely, ABC is the polar triangle of $A'B'C'$.



(*h*). In two polar triangles, each angle of one is measured y the supplement of the corresponding side of the other.

That is,

$$A = 180° - a', \qquad B = 180° - b', \qquad C = 180° - c',$$

$$A' = 180° - a, \qquad B' = 180° - b, \qquad C' = 180° - c.$$

(*i*). Either angle of a spherical triangle is greater than the ifference between 180° and the sum of the other two angles.

187. A spherical triangle is called *right*, *bi-rectangular*, or *i-rectangular*, according as it has one, two, or three right ngles.

It is called *quadrantal* when it has one side equal to a uadrant.

It is evident from Art. 186, (*h*), that the polar triangle of right spherical triangle is a quadrantal triangle.

XIV. GENERAL FORMULÆ FOR SPHERICAL TRIANGLES.

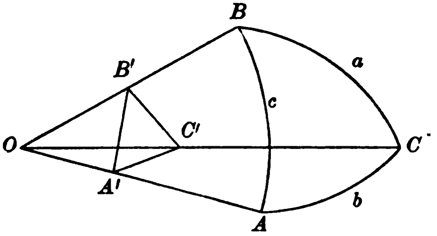
188. *In any spherical triangle the cosine of either side is equal to the product of the cosines of the other sides, plus the continued product of their sines and the cosine of their included angle.*

For example,

$$\cos a = \cos b \cos c + \sin b \sin c \cos A$$

To prove this we may distinguish five cases, in each of which A and a may have any values between 0° and 180°.

CASE I. *When b and c are both less than* 90°.



Let ABC be a spherical triangle, and O the centre of the sphere. Join OA, OB, OC; then, by Art. 181, the angles BOC, COA, and AOB are the measures of the sides a, b, and c, respectively. At any point A' in OA draw $A'B'$ and $A'C'$ perpendicular to OA in the planes OAB and OAC, respectively, and join $B'C'$. Since by hypothesis b and c are both less than 90°, the angles COA and AOB are both acute, and the perpendiculars will meet the edges OB and OC if sufficiently produced. By Art. 182 the angle $B'A'C'$ is equal to the angle A of the spherical triangle.

In the triangles $OB'C'$ and $A'B'C'$, by Art. 146,

$$B'C'^2 = OB'^2 + OC'^2 - 2\,OB' \cdot OC' \cos a$$

$$B'C'^2 = A'B'^2 + A'C'^2 - 2\,A'B' \cdot A'C' \cos A$$

Subtracting these equations, we have

$$0 = OB'^2 - A'B'^2 + OC'^2 - A'C'^2$$
$$- 2\,OB' \cdot OC' \cos a + 2\,A'B' \cdot A'C' \cos A$$

But, in the right triangles $OA'B'$ and $OA'C'$, by Geometry,

$$OB'^2 - A'B'^2 = OA'^2, \text{ and } OC'^2 - A'C'^2 = OA'^2.$$

Hence, transposing,

$$2\,OB' \cdot OC' \cos a = 2\,OA'^2 + 2\,A'B' \cdot A'C' \cos A$$

r,
$$\cos a = \frac{OA'}{OB'} \cdot \frac{OA'}{OC'} + \frac{A'B'}{OB'} \cdot \frac{A'C'}{OC'} \cos A$$

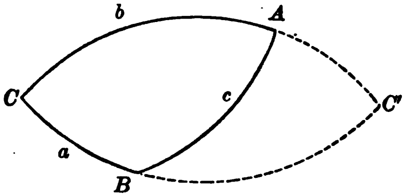
But, in the right triangles $OA'B'$ and $OA'C'$, we have

$$\frac{OA'}{OB'} = \cos c, \quad \frac{OA'}{OC'} = \cos b, \quad \frac{A'B'}{OB'} = \sin c, \quad \frac{A'C'}{OC'} = \sin b.$$

Substituting,

$$\cos a = \cos b \cos c + \sin b \sin c \cos A.$$

Case II. *When one of the sides b and c is greater than* 90° *nd the other is less than* 90°.



In the triangle ABC, let b be greater than 90°, and c less ian 90°. Complete the lune $ACBC'$. Then, in the trian- le ABC', the side $AC' = 180° - b$, and is consequently ss than 90°; that is, the two sides about the angle BAC' 'e less than 90°; hence, by Case I.,

$$\cos BC' = \cos AC' \cos c + \sin AC' \sin c \cos BAC'$$

But,

$BC' = 180° - a$, $AC' = 180° - b$, and $BAC' = 180° - A$.

Hence,

$$\cos(180° - a) = \cos(180° - b)\cos c + \sin(180° - b)\sin c\cos(180° - A)$$

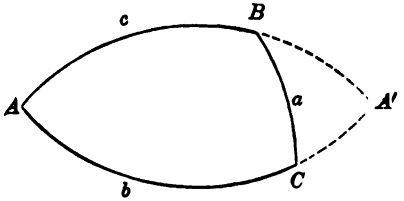
or (Art. 43),

$$-\cos a = (-\cos b)\cos c + \sin b\sin c\,(-\cos A)$$

That is, $\cos a = \cos b\cos c + \sin b\sin c\cos A$

Similarly, we may prove the theorem when b is less than 90°, and c greater than 90°.

CASE III. *When b and c are both greater than* 90°.



In the triangle ABC let b and c be both greater than 90°. Complete the lune $ABA'C$. Then, in the triangle $A'BC$, $CA' = 180° - b$, and $BA' = 180° - c$; that is, both sides about the angle A' are less than 90°; hence, by Case I.,

$$\cos a = \cos CA'\cos BA' + \sin CA'\sin BA'\cos A'$$

But $A' = A$; hence,

$$\cos a = \cos(180° - b)\cos(180° - c) + \sin(180° - b)\sin(180° - c)\cos A$$

or (Art. 43),

$$\cos a = (-\cos b)(-\cos c) + \sin b\sin c\cos A$$

That is,

$$\cos a = \cos b\cos c + \sin b\sin c\cos A$$

Case IV. ***When one of the sides b and c is equal to*** 90°.

For example, let $b = 90°$. The general formula for $\cos a$ comes in this case

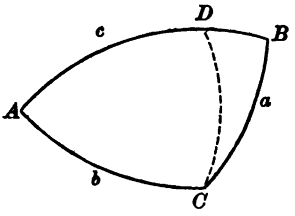
$$\cos a = 0 \times \cos c + 1 \times \sin c \cos A$$

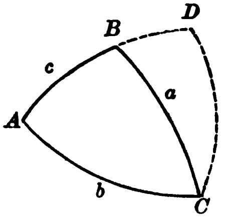
,

$$\cos a = \sin c \cos A$$

We must then prove that, in a spherical triangle where b is °, the above relation is true.

There will be two cases, according as c is greater or less an 90°.





In each figure let the side $b = 90°$. Lay off upon the arc B the distance $AD = 90°$, and draw the arc CD. Then, Geometry, the arc CD is the measure of the angle A.

If $A = 90°$, that is, if $CD = 90°$, C will be the pole of the le c, since the points A and D are each at a quadrant's stance from C. Consequently, a will be equal to 90°, and ır formula becomes

$$\cos 90° = \sin c \cos 90°$$

,

$$0 = 0$$

hich is identical.

If A is not equal to 90°, neither side about the angle BDC equal to 90°, and we have, by Cases I. and II.,

$$\cos a = \cos CD \cos BD + \sin CD \sin BD \cos CDB$$

But $CD = A$; $BD = c - 90°$ in the first figure, and $90° - c$ the second; and $CDB = 90°$. Hence, since $\cos CDB = 0$, have

$$\cos a = \cos A \cos (c - 90°), \text{ or } \cos A \cos (90° - c)$$

In either case, $\cos a = \sin c \cos A$

which was to be proved.

In a similar manner we may prove the theorem to hold when $c = 90°$.

Case V. *When b and c are both equal to* 90°.

In this case, the general formula for $\cos a$ becomes

$$\cos a = 0 \times 0 + 1 \times 1 \times \cos A$$

or, $\cos a = \cos A$

We must then prove that, in a spherical triangle where b and c are both quadrants, the above relation is true.

By Geometry, if the sides CA and AB are both quadrants, the side BC is the measure of the angle A; that is, $a = A$.

Hence, $\cos a = \cos A$

which was to be proved.

Thus, the relation

$$\cos a = \cos b \cos c + \sin b \sin c \cos A \qquad \textbf{(131)}$$

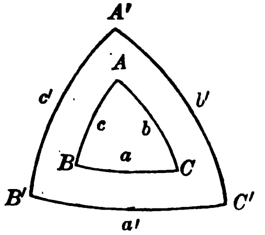
has been proved to hold universally. In a similar manner we may prove

$$\cos b = \cos c \cos a + \sin c \sin a \cos B \qquad \textbf{(132)}$$

$$\cos c = \cos a \cos b + \sin a \sin b \cos C \qquad \textbf{(133)}$$

189. The results of the preceding article may be considered as the fundamental formulæ of Spherical Trigonometry, since all other trigonometrical relations between the sides and angles of a spherical triangle can be derived from them.

190. Let ABC and $A'B'C'$ be a pair of polar triangles.



Applying the theorem of Art. 188 to the side a' of the trigle $A'B'C''$, we have

$$\cos a' = \cos b' \cos c' + \sin b' \sin c' \cos A'$$

Hence, by Art. 186, (h),

$$\cos(180^\circ - A) = \cos(180^\circ - B)\cos(180^\circ - C) \\ + \sin(180^\circ - B)\sin(180^\circ - C)\cos(180^\circ - a)$$

Whence (Art. 43),

$$-\cos A = (-\cos B)(-\cos C) + \sin B \sin C(-\cos a)$$

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a \qquad (134)$$

Similarly, we may prove

$$\cos B = -\cos C \cos A + \sin C \sin A \cos b \qquad (135)$$

$$\cos C = -\cos A \cos B + \sin A \sin B \cos c \qquad (136)$$

The proofs of these formulæ illustrate a very important plication of the theory of polar triangles to Spherical igonometry. When a certain relation between the elents of a triangle has been demonstrated, an analogous ation may be at once derived from it, in which each side angle is replaced by its opposite angle or side, with itable modifications in the algebraic signs.

191. From Art. 188, we have

$$\sin b \sin c \cos A = \cos a - \cos b \cos c$$

Whence, $$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c} \qquad (137)$$

Similarly,

$$\cos B = \frac{\cos b - \cos c \cos a}{\sin c \sin a}, \quad \cos C = \frac{\cos c - \cos a \cos b}{\sin a \sin b} \qquad (138)$$

192. *The sines of the sides of a spherical triangle are in the same ratio as the sines of their opposite angles.*

$$\sin^2 A = 1 - \cos^2 A = 1 - \left(\frac{\cos a - \cos b \cos c}{\sin b \sin c}\right)^2 \qquad \text{(Art. 191)}$$

$$= \frac{\sin^2 b \sin^2 c - (\cos a - \cos b \cos c)^2}{\sin^2 b \sin^2 c}$$

$$= \frac{(1 - \cos^2 b)(1 - \cos^2 c) - (\cos a - \cos b \cos c)^2}{\sin^2 b \sin^2 c}$$

$$= \frac{1 - \cos^2 b - \cos^2 c - \cos^2 a + 2 \cos a \cos b \cos c}{\sin^2 b \sin^2 c}$$

Hence,

$$\sin A = \frac{\sqrt{[1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c]}}{\sin b \sin c} \qquad (139)$$

The *positive* sign must be taken for the radical here; for, since A, b, and c lie between 0° and 180°, their sines are positive.

Thus,

$$\frac{\sin A}{\sin a} = \frac{\sqrt{[1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c]}}{\sin a \sin b \sin c}$$

We should obtain the same value for the expressions $\frac{\sin B}{\sin b}$ and $\frac{\sin C}{\sin c}$, as is evident from the symmetry of the result.

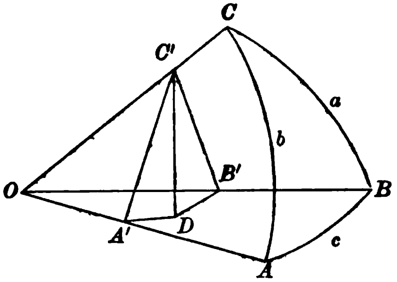
Hence, $$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c} \qquad (140)$$

This result may also be written in the forms,

$$\sin A : \sin a = \sin B : \sin b = \sin C : \sin c \qquad (141)$$

$$\sin A : \sin B : \sin C = \sin a : \sin b : \sin c \qquad (142)$$

193. The theorem of the preceding article may be proved dependently of Art. 191 as follows:



Let ABC be a spherical triangle, and O the centre of the here. Join OA, OB, and OC; then the angles BOC and OA measure the sides a and b, respectively.

At any point C' in OC draw $C'D$ perpendicular to the ane AOB. Draw DA' and DB' perpendicular to OA and B, respectively, and join $A'C'$ and $B'C'$. By Geometry, $'C'$ is perpendicular to OA, and $B'C'$ to OB. Hence, the gles $C'A'D$ and $C'B'D$ are equal to the angles A and B, spectively. (Art. 182.)

In the right triangles $OB'C'$, $OA'C'$, $A'DC'$, and $B'DC'$,

$$\sin A = \sin C'A'D = \frac{C'D}{A'C'}, \quad \sin B = \sin C'B'D = \frac{C'D}{B'C'},$$

$$\sin a = \sin B'OC' = \frac{B'C'}{OC'}, \quad \sin b = \sin A'OC' = \frac{A'C'}{OC'}.$$

Whence, $$\frac{\sin A}{\sin a} = \frac{C'D}{A'C'} \div \frac{B'C'}{OC'} = \frac{C'D \times OC'}{A'C' \times B'C'}$$

d, $$\frac{\sin B}{\sin b} = \frac{C'D}{B'C'} \div \frac{A'C'}{OC'} = \frac{C'D \times OC'}{A'C' \times B'C'}$$

Hence, $$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b}.$$

In a similar manner it can be proved that each of the above fractions is equal to $\frac{\sin C}{\sin c}$.

In the figure we have supposed each of the elements a, b, A, and B to be less than 90°; to prove the theorem universally, we should have to consider all the possible cases which could arise. On account of the inconvenience of demonstrating all these cases by the method of this article, the proof of Art. 192 is preferable, since it depends on a theorem which has been proved to hold universally.

194. Applying the method of Art. 190 to the formulæ of Art. 192, we have

$$\frac{\sin A'}{\sin a'}=\frac{\sin B'}{\sin b'}=\frac{\sin C'}{\sin c'}$$

or, by Art. 186, (*h*),

$$\frac{\sin(180^\circ-a)}{\sin(180^\circ-A)}=\frac{\sin(180^\circ-b)}{\sin(180^\circ-B)}=\frac{\sin(180^\circ-c)}{\sin(180^\circ-C)}$$

That is,

$$\frac{\sin a}{\sin A}=\frac{\sin b}{\sin B}=\frac{\sin c}{\sin C} \qquad \text{(Art. 43)}$$

Thus no new relations are obtained in this way.

195. By Art. 188,

$$\cos b=\cos c\cos a+\sin c\sin a\cos B \qquad (\text{A})$$

$$\cos c=\cos a\cos b+\sin a\sin b\cos C \qquad (\text{B})$$

Also (Art. 192),

$$\frac{\sin c}{\sin C}=\frac{\sin b}{\sin B},\ \text{or}\ \sin c=\frac{\sin b\sin C}{\sin B} \qquad (\text{C})$$

Substituting from (B) and (C) in (A),

$$\begin{aligned}\cos b &= \cos a\,(\cos a\cos b+\sin a\sin b\cos C)\\ &\qquad +\frac{\sin b\sin C}{\sin B}\sin a\cos B\\ &= \cos b\cos^2 a+\sin a\cos a\sin b\cos C\\ &\qquad +\sin a\sin b\cot B\sin C\end{aligned}$$

$$\therefore \cos b\,(1-\cos^2 a)=\sin a\,\sin b\,(\sin C\cot B+\cos a\cos C)$$

r, $$\cos b\,\sin^2 a=\sin a\,\sin b\,(\sin C\cot B+\cos a\cos C)$$

Dividing through by $\sin a \sin b$, and remembering that $\frac{\text{os}\, b}{\text{in}\, b}=\cot b$, we obtain

$$\sin a\cot b=\sin C\cot B+\cos a\cos C \qquad \textbf{(143)}$$

Similarly, we may prove,

$$\sin b\cot c=\sin A\cot C+\cos b\cos A \qquad \textbf{(144)}$$
$$\sin c\cot a=\sin B\cot A+\cos c\cos B \qquad \textbf{(145)}$$
$$\sin a\cot c=\sin B\cot C+\cos a\cos B \qquad \textbf{(146)}$$
$$\sin b\cot a=\sin C\cot A+\cos b\cos C \qquad \textbf{(147)}$$
$$\sin c\cot b=\sin A\cot B+\cos c\cos A \qquad \textbf{(148)}$$

The last three formulæ may also be derived from the first hree by the method of Art. 190.

196. By Art. 191,

$$1-\cos A=1-\frac{\cos a-\cos b\cos c}{\sin b\sin c}$$

$$=\frac{\cos b\cos c+\sin b\sin c-\cos a}{\sin b\sin c}$$

r (Art. 69), $$2\sin^2\tfrac{1}{2}A=\frac{\cos(b-c)-\cos a}{\sin b\sin c}$$

But (Art. 72),

$$\cos y-\cos x=2\sin\tfrac{1}{2}(x+y)\sin\tfrac{1}{2}(x-y)$$

Hence,

$$2\sin^2\tfrac{1}{2}A=\frac{2\sin\frac{1}{2}\{a+(b-c)\}\sin\frac{1}{2}\{a-(b-c)\}}{\sin b\sin c}$$

r, $$\sin^2\tfrac{1}{2}A=\frac{\sin\frac{1}{2}(a+b-c)\sin\frac{1}{2}(a-b+c)}{\sin b\sin c}$$

Let now

$$a+b+c=2s$$

Then $a+b-c=a+b+c-2c=2s-2c=2(s-c)$

nd $a-b+c=a+b+c-2b=2s-2b=2(s-b)$

Substituting, we have

$$\sin^2 \tfrac{1}{2} A = \frac{\sin(s-b)\sin(s-c)}{\sin b \sin c}$$

or,
$$\sin \tfrac{1}{2} A = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin b \sin c}} \qquad (149)$$

Similarly, we may prove

$$\left.\begin{aligned} \sin \tfrac{1}{2} B &= \sqrt{\frac{\sin(s-c)\sin(s-a)}{\sin c \sin a}} \\ \sin \tfrac{1}{2} C &= \sqrt{\frac{\sin(s-a)\sin(s-b)}{\sin a \sin b}} \end{aligned}\right\} \qquad (150)$$

197. By Art. 191,

$$1 + \cos A = 1 + \frac{\cos a - \cos b \cos c}{\sin b \sin c}$$

$$= \frac{\cos a - (\cos b \cos c - \sin b \sin c)}{\sin b \sin c}$$

or,
$$2\cos^2 \tfrac{1}{2} A = \frac{\cos a - \cos(b+c)}{\sin b \sin c}$$

Hence, as in the proof in Art. 196,

$$\cos^2 \tfrac{1}{2} A = \frac{\sin \frac{1}{2}(b+c+a) \sin \frac{1}{2}(b+c-a)}{\sin b \sin c}$$

Letting $a+b+c=2s$, we have

$$b+c-a = a+b+c-2a = 2(s-a)$$

Substituting,

$$\cos^2 \tfrac{1}{2} A = \frac{\sin s \sin(s-a)}{\sin b \sin c}$$

or,
$$\cos \tfrac{1}{2} A = \sqrt{\frac{\sin s \sin(s-a)}{\sin b \sin c}} \qquad (151)$$

Similarly, we may prove

$$\left.\begin{aligned} \cos \tfrac{1}{2} B &= \sqrt{\frac{\sin s \sin(s-b)}{\sin c \sin a}} \\ \cos \tfrac{1}{2} C &= \sqrt{\frac{\sin s \sin(s-c)}{\sin a \sin b}} \end{aligned}\right\} \qquad (152)$$

198. By Arts. 196 and 197.

$$\tan\tfrac{1}{2}A = \frac{\sin\frac{1}{2}A}{\cos\frac{1}{2}A} = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin b\sin c}} \div \sqrt{\frac{\sin s\sin(s-a)}{\sin b\sin c}}$$

$$= \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s\sin(s-a)}} \qquad (153)$$

Similarly,

$$\left.\begin{aligned}\tan\tfrac{1}{2}B &= \sqrt{\frac{\sin(s-c)\sin(s-a)}{\sin s\sin(s-b)}}\\ \tan\tfrac{1}{2}C &= \sqrt{\frac{\sin(s-a)\sin(s-b)}{\sin s\sin(s-c)}}\end{aligned}\right\} \qquad (154)$$

Note. The radicals are taken *positive* in Arts. 196, 197, and 198, since $\frac{1}{2}A$, $\frac{1}{2}B$, and $\frac{1}{2}C$ are acute angles.

199. By Art. 190,

$$\sin B\sin C\cos a = \cos A + \cos B\cos C$$

or,

$$\cos a = \frac{\cos A + \cos B\cos C}{\sin B\sin C}$$

Whence,

$$1 - \cos a = 1 - \frac{\cos A + \cos B\cos C}{\sin B\sin C}$$

That is (Art. 69),

$$2\sin^2\tfrac{1}{2}a = \frac{-\cos B\cos C + \sin B\sin C - \cos A}{\sin B\sin C}$$

$$= -\frac{\cos(B+C) + \cos A}{\sin B\sin C}$$

But, by Art. 72,

$$\cos x + \cos y = 2\cos\tfrac{1}{2}(x+y)\cos\tfrac{1}{2}(x-y)$$

Hence, $\sin^2\frac{1}{2}a = -\dfrac{\cos\frac{1}{2}(B+C+A)\cos\frac{1}{2}(B+C-A)}{\sin B\sin C}$

Letting $A+B+C=2S$

$$B+C-A = A+B+C-2A = 2(S-A)$$

Substituting,

$$\sin^2\tfrac{1}{2}a = -\frac{\cos S\cos(S-A)}{\sin B\sin C}$$

or, $$\sin\tfrac{1}{2}a=\sqrt{-\frac{\cos S\cos(S-A)}{\sin B\sin C}} \tag{155}$$

Similarly, we may prove

$$\left.\begin{aligned}\sin\tfrac{1}{2}b&=\sqrt{-\frac{\cos S\cos(S-B)}{\sin C\sin A}}\\ \sin\tfrac{1}{2}c&=\sqrt{-\frac{\cos S\cos(S-C)}{\sin A\sin B}}\end{aligned}\right\} \tag{156}$$

200. By the first part of the preceding article,

$$1+\cos a=1+\frac{\cos A+\cos B\cos C}{\sin B\sin C}$$

$$=\frac{\cos A+\cos B\cos C+\sin B\sin C}{\sin B\sin C}$$

or, $$2\cos^2\tfrac{1}{2}a=\frac{\cos A+\cos(B-C)}{\sin B\sin C}$$

Whence, $$\cos^2\tfrac{1}{2}a=\frac{\cos\frac{1}{2}[A+B-C]\cos\frac{1}{2}[A-(B-C)]}{\sin B\sin C}$$

$$=\frac{\cos\frac{1}{2}(A+B-C)\cos\frac{1}{2}(A-B+C)}{\sin B\sin C}$$

But $A+B-C=2(S-C)$, and $A-B+C=2(S-B)$

Hence, $$\cos^2\tfrac{1}{2}a=\frac{\cos(S-C)\cos(S-B)}{\sin B\sin C}$$

or, $$\cos\tfrac{1}{2}a=\sqrt{\frac{\cos(S-B)\cos(S-C)}{\sin B\sin C}} \tag{157}$$

Similarly,

$$\left.\begin{aligned}\cos\tfrac{1}{2}b&=\sqrt{\frac{\cos(S-C)\cos(S-A)}{\sin C\sin A}}\\ \cos\tfrac{1}{2}c&=\sqrt{\frac{\cos(S-A)\cos(S-B)}{\sin A\sin B}}\end{aligned}\right\} \tag{158}$$

201. By Arts. 199 and 200,

$$\tan\tfrac{1}{2}a=\frac{\sin\frac{1}{2}a}{\cos\frac{1}{2}a}=\sqrt{-\frac{\cos S\cos(S-A)}{\cos(S-B)\cos(S-C)}} \tag{159}$$

Similarly,

$$\left.\begin{aligned}\tan\tfrac{1}{2}b &= \sqrt{-\frac{\cos S\cos(S-B)}{\cos(S-C)\cos(S-A)}}\\ \tan\tfrac{1}{2}c &= \sqrt{-\frac{\cos S\cos(S-C)}{\cos(S-A)\cos(S-B)}}\end{aligned}\right\}\qquad(160)$$

202. That the values of the sines, cosines, and tangents of the half sides given in Arts. 199, 200, and 201 are *real*, may be shown as follows:

By Art. 186, (*f*),

$$A+B+C>180^\circ \text{ and } <540^\circ$$

Hence, S or $\tfrac{1}{2}(A+B+C)>90^\circ$ and $<270^\circ$

Consequently $\cos S$ is negative.

Again, by Art. 186, (*a*), in the polar triangle of ABC, we shall have

$$a'<b'+c'$$

Whence, (Art. 186, (*h*)),

$$180^\circ-A<180^\circ-B+180^\circ-C$$

or, $$B+C-A<180^\circ$$

Therefore,

$$S-A \text{ or } \tfrac{1}{2}(B+C-A)<90^\circ$$

Also, $B+C>0$; and as $A<180^\circ$, $-A>-180^\circ$

Whence, by addition,

$$B+C-A>-180^\circ$$

Therefore, $$S-A>-90^\circ$$

That is, $S-A$ lies between 90° and -90°, and consequently its cosine is positive; and, similarly, we may show that $\cos(S-B)$ and $\cos(S-C)$ are positive.

Thus the values of the functions of the half sides are real.

The formulæ of Arts. 199, 200, and 201 may be derived in another manner by applying the principles of Art. 190 to the formulæ of Arts. 196, 197, and 198.

NAPIER'S ANALOGIES.

203. By Art. 198,

$$\frac{\tan\frac{1}{2}A}{\tan\frac{1}{2}B}=\sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s\sin(s-a)}}\div\sqrt{\frac{\sin(s-c)\sin(s-a)}{\sin s\sin(s-b)}}$$

$$=\sqrt{\frac{\sin^2(s-b)}{\sin^2(s-a)}}=\frac{\sin(s-b)}{\sin(s-a)}$$

Whence, by composition and division,

$$\frac{\tan\frac{1}{2}A+\tan\frac{1}{2}B}{\tan\frac{1}{2}A-\tan\frac{1}{2}B}=\frac{\sin(s-b)+\sin(s-a)}{\sin(s-b)-\sin(s-a)}$$

From this, by Arts. 65 and 73,

$$\frac{\sin(\frac{1}{2}A+\frac{1}{2}B)}{\sin(\frac{1}{2}A-\frac{1}{2}B)}=\frac{\tan\frac{1}{2}[s-b+(s-a)]}{\tan\frac{1}{2}[s-b-(s-a)]}$$

or,

$$\frac{\sin\frac{1}{2}(A+B)}{\sin\frac{1}{2}(A-B)}=\frac{\tan\frac{1}{2}(2s-a-b)}{\tan\frac{1}{2}(a-b)}$$

But, $2s-a-b=a+b+c-a-b=c$; hence,

$$\frac{\sin\frac{1}{2}(A+B)}{\sin\frac{1}{2}(A-B)}=\frac{\tan\frac{1}{2}c}{\tan\frac{1}{2}(a-b)}\tag{161}$$

204. By Art. 198,

$$\tan\tfrac{1}{2}A\tan\tfrac{1}{2}B=\sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s\sin(s-a)}}\sqrt{\frac{\sin(s-c)\sin(s-a)}{\sin s\sin(s-b)}}$$

$$=\sqrt{\frac{\sin^2(s-c)}{\sin^2 s}}=\frac{\sin(s-c)}{\sin s}$$

Whence,

$$\frac{1-\tan\frac{1}{2}A\tan\frac{1}{2}B}{1+\tan\frac{1}{2}A\tan\frac{1}{2}B}=\frac{1-\dfrac{\sin(s-c)}{\sin s}}{1+\dfrac{\sin(s-c)}{\sin s}}$$

$$=\frac{\sin s-\sin(s-c)}{\sin s+\sin(s-c)}$$

From this, by Arts. 65 and 73,

$$\frac{\cos(\frac{1}{2}A+\frac{1}{2}B)}{\cos(\frac{1}{2}A-\frac{1}{2}B)}=\frac{\tan\frac{1}{2}[s-(s-c)]}{\tan\frac{1}{2}[s+(s-c)]}$$

But, $2s-c=a+b+c-c=a+b$; hence,

$$\frac{\cos\frac{1}{2}(A+B)}{\cos\frac{1}{2}(A-B)}=\frac{\tan\frac{1}{2}c}{\tan\frac{1}{2}(a+b)} \qquad (162)$$

205. Proceeding according to the method of Art. 190, we have, by Art. 203,

$$\frac{\sin\frac{1}{2}(A'+B')}{\sin\frac{1}{2}(A'-B')}=\frac{\tan\frac{1}{2}c'}{\tan\frac{1}{2}(a'-b')}$$

But,

$$\begin{aligned}\tfrac{1}{2}(A'+B')&=\tfrac{1}{2}(180^\circ-a+180^\circ-b)=180^\circ-\tfrac{1}{2}(a+b)\\ \tfrac{1}{2}(A'-B')&=\tfrac{1}{2}(180^\circ-a-180^\circ+b)=\tfrac{1}{2}(-a+b)\\ \tfrac{1}{2}c'&=\tfrac{1}{2}(180^\circ-C)\qquad\qquad=90^\circ-\tfrac{1}{2}C\\ \tfrac{1}{2}(a'-b')&=\tfrac{1}{2}(180^\circ-A-180^\circ+B)=\tfrac{1}{2}(-A+B)\end{aligned}$$

Hence, $\dfrac{\sin[180^\circ-\frac{1}{2}(a+b)]}{\sin\frac{1}{2}(-a+b)}=\dfrac{\tan(90^\circ-\frac{1}{2}C)}{\tan\frac{1}{2}(-A+B)}$

Whence, by Arts. 40, 42, 43,

$$\frac{\sin\frac{1}{2}(a+b)}{-\sin\frac{1}{2}(a-b)}=\frac{\cot\frac{1}{2}C}{-\tan\frac{1}{2}(A-B)}$$

or,

$$\frac{\sin\frac{1}{2}(a+b)}{\sin\frac{1}{2}(a-b)}=\frac{\cot\frac{1}{2}C}{\tan\frac{1}{2}(A-B)} \qquad (163)$$

Similarly, from Art. 204, we have

$$\frac{\cos\frac{1}{2}(A'+B')}{\cos\frac{1}{2}(A'-B')}=\frac{\tan\frac{1}{2}c'}{\tan\frac{1}{2}(a'+b')}$$

That is,

$$\frac{\cos[180^\circ-\frac{1}{2}(a+b)]}{\cos\frac{1}{2}(-a+b)}=\frac{\tan(90^\circ-\frac{1}{2}C)}{\tan[180^\circ-\frac{1}{2}(A+B)]}$$

Whence, $\dfrac{-\cos\frac{1}{2}(a+b)}{\cos\frac{1}{2}(a-b)}=\dfrac{\cot\frac{1}{2}C}{-\tan\frac{1}{2}(A+B)}$

or,
$$\frac{\cos\frac{1}{2}(a+b)}{\cos\frac{1}{2}(a-b)}=\frac{\cot\frac{1}{2}C}{\tan\frac{1}{2}(A+B)} \qquad (164)$$

206. The theorems exemplified by the results of the last three articles are known as *Napier's Analogies;* there will be other forms in each case, according as other elements of the triangle are used.

The formulæ of Art. 205 might have been proved independently, as in Arts. 203 and 204, by starting with the results of Art. 201.

DELAMBRE'S ANALOGIES.

207. $\sin\frac{1}{2}(A\pm B)=\sin\left(\frac{A}{2}\pm\frac{B}{2}\right)$

$$=\sin\frac{A}{2}\cos\frac{B}{2}\pm\sin\frac{B}{2}\cos\frac{A}{2}$$

Substituting from Arts. 196 and 197,

$$\sin\tfrac{1}{2}(A\pm B)=\sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin b\sin c}}\sqrt{\frac{\sin s\sin(s-b)}{\sin c\sin a}}$$
$$\pm\sqrt{\frac{\sin(s-c)\sin(s-a)}{\sin c\sin a}}\sqrt{\frac{\sin s\sin(s-a)}{\sin b\sin c}}$$
$$=\frac{\sin(s-b)\pm\sin(s-a)}{\sin c}\sqrt{\frac{\sin s\sin(s-c)}{\sin a\sin b}}$$

Whence, by Arts. 72 and 197,

$$\sin\tfrac{1}{2}(A+B)=\frac{2\sin\frac{1}{2}[s-b+s-a]\cos\frac{1}{2}[s-b-(s-a)]}{\sin c}\cos\tfrac{1}{2}C$$

$$\sin\tfrac{1}{2}(A-B)=\frac{2\cos\frac{1}{2}[s-b+s-a]\sin\frac{1}{2}[s-b-(s-a)]}{\sin c}\cos\tfrac{1}{2}C$$

But $2s-a-b=a+b+c-a-b=c;$
also (Art. 67), $\sin c=2\sin\frac{1}{2}c\cos\frac{1}{2}c.$

Hence,
$$\sin\tfrac{1}{2}(A+B)=\frac{2\sin\frac{1}{2}c\cos\frac{1}{2}(a-b)}{2\sin\frac{1}{2}c\cos\frac{1}{2}c}\cos\tfrac{1}{2}C$$
$$=\frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}c}\cos\tfrac{1}{2}C \qquad (165)$$

$$\sin\tfrac{1}{2}(A-B)=\frac{2\cos\frac{1}{2}c\sin\frac{1}{2}(a-b)}{2\sin\frac{1}{2}c\cos\frac{1}{2}c}\cos\tfrac{1}{2}C$$

$$=\frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}c}\cos\tfrac{1}{2}C \qquad (166)$$

208. Since $\tan\frac{1}{2}(A+B)=\frac{\sin\frac{1}{2}(A+B)}{\cos\frac{1}{2}(A+B)}$, the last equation of Art. 205 can be written

$$\tan\tfrac{1}{2}C\frac{\cos\frac{1}{2}(a+b)}{\cos\frac{1}{2}(a-b)}=\frac{\cos\frac{1}{2}(A+B)}{\sin\frac{1}{2}(A+B)}$$

or, $$\cos\tfrac{1}{2}(A+B)=\frac{\cos\frac{1}{2}(a+b)}{\cos\frac{1}{2}(a-b)}\tan\tfrac{1}{2}C\sin\tfrac{1}{2}(A+B)$$

Substituting the value of $\sin\frac{1}{2}(A+B)$ from Art. 207,

$$\cos\tfrac{1}{2}(A+B)=\frac{\cos\frac{1}{2}(a+b)}{\cos\frac{1}{2}(a-b)}\tan\tfrac{1}{2}C\frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}c}\cos\tfrac{1}{2}C$$

$$=\frac{\cos\frac{1}{2}(a+b)}{\cos\frac{1}{2}c}\sin\tfrac{1}{2}C \qquad (167)$$

Similarly, we may prove

$$\cos\tfrac{1}{2}(A-B)=\frac{\sin\frac{1}{2}(a+b)}{\sin\frac{1}{2}c}\sin\tfrac{1}{2}C \qquad (168)$$

209. The theorems of which the last two articles are examples are called *Delambre's Analogies;* and in each case there will be other forms, according as other sides and angles are employed.

The formulæ of Art. 208 might have been derived in the same manner as those of Art. 207; they may also be proved in still another way, by applying the method of Art. 190 to the results of Art. 207.

210. Certain properties of spherical triangles are readily proved by aid of the theorems of this chapter.

(*a*). *If a side differs more from* 90° *than another side, it is in the same quadrant as its opposite angle.*

For example, if a differs more from 90° than b, a and A are in the same quadrant.

To prove this, we have, by Art. 191,

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}$$

Since b and c are both less than 180°, $\sin b$ and $\sin c$ are positive; and, since a differs more from 90° than b, $\cos a$ is numerically greater than $\cos b$. But, $\cos c$ is less than unity; hence, $\cos a$ is numerically greater than $\cos b \cos c$. Hence, the sign of the fraction will be the same as the sign of $\cos a$; that is, $\cos A$ and $\cos a$ are both positive, or both negative; hence, A and a are both less than 90°, or both greater than 90°.

(b). *If an angle differs more from 90° than another angle, it is in the same quadrant as its opposite side.*

For example, if A differs more from 90° than B, A and a are in the same quadrant.

To prove this, we have, by Art. 190,

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a$$

Whence, $$\cos a = \frac{\cos A + \cos B \cos C}{\sin B \sin C}$$

We may show, as before, that $\sin B$ and $\sin C$ are positive, and that $\cos A$ is numerically greater than $\cos B \cos C$; hence, as in the previous case, A and a must be both less than 90°, or both greater than 90°.

(c). *The half sum of any two sides is in the same quadrant as the half sum of the opposite angles.*

For example, $\frac{1}{2}(a+b)$ is less than, equal to, or greater than 90°, according as $\frac{1}{2}(A+B)$ is less than, equal to, or greater than 90°.

To prove this, we have, by Art. 204,

$$\frac{\cos\frac{1}{2}(A+B)}{\cos\frac{1}{2}(A-B)}=\frac{\tan\frac{1}{2}c}{\tan\frac{1}{2}(a+b)}$$

Since, A, B, and c are each less than 180°, $\frac{1}{2}c$ is less than 90°, and $\frac{1}{2}(A-B)$ lies between 90° and $-90°$; hence, $\tan\frac{1}{2}c$ and $\cos\frac{1}{2}(A-B)$ are both positive. Thus, $\cos\frac{1}{2}(A+B)$ and $\tan\frac{1}{2}(a+b)$ must be both positive, or both negative, or else $\cos\frac{1}{2}(A+B)=0$, and $\tan\frac{1}{2}(a+b)=\infty$. That is, $\frac{1}{2}(A+B)$ and $\frac{1}{2}(a+b)$ must be both less than 90°, both greater than 90°, or both equal to 90°.

(*d*). In any triangle (which is not isosceles or equilateral) there will be in general one side which is nearer 90° than either of the others; hence, by (*a*), the other two sides must lie in the same quadrants as their opposite angles.

In an isosceles triangle, by (*c*), the half sum of the equal sides is in the same quadrant as the half sum of the equal angles; that is, either of the equal sides lies in the same quadrant as its opposite angle.

In an equilateral triangle, by the principle just proved, either side lies in the same quadrant as its opposite angle.

XV. SOLUTION OF SPHERICAL RIGHT TRIANGLES.

211. A spherical triangle is, in general, completely determined when any three of its six elements are known, even in the case where the given parts are the three angles. Thus, a right spherical triangle can be solved when any two parts besides the right angle are given.

We will now derive, from the general formulæ of the preceding chapter, the simpler forms adapted to the solution of right triangles. We shall suppose C the right angle.

212. From Art. 188,

$$\cos c = \cos a \cos b + \sin a \sin b \cos C$$

Making $C = 90°$, whence $\cos C = 0$, we have

$$\cos c = \cos a \cos b \tag{169}$$

213. From Art. 192,

$$\frac{\sin A}{\sin C} = \frac{\sin a}{\sin c}, \quad \text{and} \quad \frac{\sin B}{\sin C} = \frac{\sin b}{\sin c}$$

Making $C = 90°$, whence $\sin C = 1$, we have

$$\sin A = \frac{\sin a}{\sin c}, \quad \text{and} \quad \sin B = \frac{\sin b}{\sin c} \tag{170}$$

214. From Art. 190,

$$\cos B = -\cos C \cos A + \sin C \sin A \cos b$$

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a$$

Making $C = 90°$, these become

$$\cos B = \sin A \cos b, \quad \text{and} \quad \cos A = \sin B \cos a$$

whence,

$$\sin A = \frac{\cos B}{\cos b}, \quad \text{and} \quad \sin B = \frac{\cos A}{\cos a} \tag{171}$$

215. From Art. 190,

$$\cos C = -\cos A \cos B + \sin A \sin B \cos c$$

Making $C = 90°$,

$$0 = -\cos A \cos B + \sin A \sin B \cos c$$

Hence,

$$\cos c = \frac{\cos A \cos B}{\sin A \sin B} = \cot A \cot B \tag{172}$$

216. From Art. 214,

$$\cos A = \sin B \cos a, \quad \text{and} \quad \cos B = \sin A \cos b$$

whence, by Arts. 212 and 213,

$$\cos A = \frac{\sin b}{\sin c} \frac{\cos c}{\cos b}, \quad \text{and} \quad \cos B = \frac{\sin a}{\sin c} \frac{\cos c}{\cos a}$$

That is, $\cos A = \dfrac{\tan b}{\tan c}$, and $\cos B = \dfrac{\tan a}{\tan c}$ **(173)**

These results may be obtained in another way by making $C = 90^\circ$ in equations (**144**) and (**146**), Art. 195.

217. By Art. 214,

$$\tan A = \frac{\sin A}{\cos A} = \frac{\sin A}{\sin B \cos a} = \frac{\sin a}{\sin b \cos a} \quad \text{(Art. 192)}$$

or, $$\left.\begin{aligned} \tan A &= \frac{\tan a}{\sin b} \\ \text{Similarly,} \quad \tan B &= \frac{\tan b}{\sin a} \end{aligned}\right\} \qquad \textbf{(174)}$$

These may also be derived from equations (**143**) and (**147**) of Art. 195.

218. The results of the last five articles are collected here for the convenience of the student.

$$\cos c = \cos a \cos b$$

$$\sin A = \frac{\sin a}{\sin c} \qquad \sin B = \frac{\sin b}{\sin c}$$

$$\cos A = \frac{\tan b}{\tan c} \qquad \cos B = \frac{\tan a}{\tan c}$$

$$\tan A = \frac{\tan a}{\sin b} \qquad \tan B = \frac{\tan b}{\sin a}$$

$$\sin A = \frac{\cos B}{\cos b} \qquad \sin B = \frac{\cos A}{\cos a}$$

$$\cos c = \cot A \cot B$$

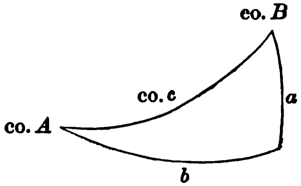
By comparing the formulæ for the sines, cosines, and tangents of the angles with the corresponding forms for plane triangles, as given in Arts. 11 and 13, no difficulty will be experienced in retaining them in the memory.

NAPIER'S RULES OF CIRCULAR PARTS.

219. These are two artificial rules which include all the formulæ of the preceding article.

In a spherical right triangle, the elements a and b, and the complements of the elements A, B, and c, which are written in an abbreviated form, co. A, co. B, and co. c, are called the *circular parts.*

If we suppose them arranged in the order in which the letters occur in the triangle,



any one of the five may be selected and called the *middle part;* the two immediately adjacent are then called the *adjacent parts*, and the remaining two the *opposite parts.*

Then, Napier's rules are:

The sine of the middle part equals the product of the tangents of the adjacent parts.

The sine of the middle part equals the product of the cosines of the opposite parts.

220. These may be proved by taking each of the circular parts in succession as the middle part, and showing that the results agree with the formulæ of Art. 218.

If a is the middle part, b and co. B are the adjacent parts, and co. c and co. A the opposite parts. Then, the rules give

$$\sin a = \tan b \tan(\text{co. } B) \qquad \sin a = \cos(\text{co. } c)\cos(\text{co. } A)$$

or (Art. 13),

$$\sin a = \tan b \cot B \qquad \sin a = \sin c \sin A$$

which agree with Arts. 217 and 213.

If b is the middle part, a and co. A are the adjacent parts, and co. c and co. B the opposite parts. Then,

$$\sin b = \tan a \tan (\text{co. } A) \qquad \sin b = \cos (\text{co. } c) \cos (\text{co. } B)$$

or, $\sin b = \tan a \cot A \qquad \sin b = \sin c \sin B$

which agree with Arts. 217 and 213.

If co. c is the middle part, co. A and co. B are the adjacent parts, and a and b the opposite parts. Then,

$$\sin (\text{co. } c) = \tan (\text{co. } A) \tan (\text{co. } B)$$
$$\sin (\text{co. } c) = \cos a \cos b$$

or, $\cos c = \cot A \cot B \qquad \cos c = \cos a \cos b$

which agree with Arts. 212 and 215.

If co. A is the middle part, b and co. c are the adjacent parts, and a and co. B the opposite parts. Then,

$$\sin (\text{co. } A) = \tan b \tan (\text{co. } c)$$
$$\sin (\text{co. } A) = \cos a \cos (\text{co. } B)$$

or, $\cos A = \tan b \cot c \qquad \cos A = \cos a \sin B$

which agree with Arts. 216 and 214.

If co. B is the middle part, a and co. c are the adjacent parts, and b and co. A the opposite parts. Then,

$$\sin (\text{co. } B) = \tan a \tan (\text{co. } c)$$
$$\sin (\text{co. } B) = \cos b \cos (\text{co. } A)$$

or, $\cos B = \tan a \cot c \qquad \cos B = \cos b \sin A$

which agree with Arts. 216 and 214.

221. Writers on Trigonometry differ as to the practical value of Napier's rules to the computer; but, in the opinion of the highest authorities, it seems to be regarded as preferable to recollect the formulæ by comparing them with the analogous forms of Plane Trigonometry, as mentioned in Art. 218.

SOLUTION OF RIGHT TRIANGLES.

222. In the solution of spherical right triangles, there will be six cases which can arise:

1. *Given the hypothenuse and an adjacent angle.*
2. *Given an angle and its opposite side.*
3. *Given an angle and its adjacent side.*
4. *Given the hypothenuse and another side.*
5. *Given the two sides a and b.*
6. *Given the two angles A and B.*

223. Any of the above cases may be solved by aid of the formulæ of Art. 218. To compute either of the remaining elements when any two are given, the proper equation is *that which involves the two given parts and the required part.*

If all the remaining elements are required, the following rule may be found convenient in selecting the proper equations:

Take the three formulæ which involve the given parts.

224. In making the computations, attention must be paid to the algebraic signs of the functions; the cosine, tangent, or cotangent of an angle greater than 90° being taken negative. It will be found convenient to put the sign of each function just above or beneath it, as illustrated in the examples of Art. 229; the sign of the first member being then determined in accordance with the principle that like signs produce plus, and unlike signs produce minus. In the examples after the first of Art. 229, the signs are omitted in every equation in which all the terms are positive.

In looking out the angles corresponding, if the function is a cosine, tangent, or cotangent, its sign determines whether the angle is less or greater than 90°; that is, if it is positive, the angle is less than 90°; and, if negative, it is greater than 90°, and the supplement of the angle obtained from the tables must be taken.

If the function is a sine, since the sine of an angle is equal to the sine of its supplement, both the acute angle obtained from the tables and its supplement must be regarded as solutions, unless the ambiguity can be removed by the principles of the next two articles.

225. *In a spherical right triangle, an angle and its opposite side are in the same quadrant.*

By Art. 218, $\sin B = \frac{\cos A}{\cos a}$.

Since $\sin B$ is always positive, $\cos A$ and $\cos a$ must be both positive or both negative. Hence, A and a must be both less than 90°, or both greater than 90°.

Similarly, from the equation $\sin A = \frac{\cos B}{\cos b}$, it follows that B and b are in the same quadrant.

226. *If the hypothenuse is less than* 90°, *the other two sides are in the same quadrant; and, if the hypothenuse is greater than* 90°, *the other two sides are in different quadrants.*

By Art. 218, $\cos c = \cos a \cos b$.

If c is less than 90°, $\cos c$ is positive; hence $\cos a$ and $\cos b$ are both positive or both negative; that is, a and b are both less than 90°, or both greater than 90°.

If c is greater than 90°, $\cos c$ is negative; hence, $\cos a$ and $\cos b$ must be of opposite sign; that is, one of the sides a and b is less than 90°, and the other is greater than 90°.

227. It is convenient in the solution to have a check on the logarithmic work, which may be done in every case without the necessity of looking out any new logarithms. Examples of this will be found in Art. 229. The check formula for any particular example may be selected from the set in Art. 218, by the application of the following rule:

Take the formula containing the three required parts.

228. If Napier's rules are used, the following rule will indicate which of the circular parts corresponding to the given elements and any required element is to be regarded as the middle part.

If these three circular parts are adjacent, take the middle one as the middle part, and the others are then adjacent parts. If they are not adjacent, take the part which is not adjacent to either of the others as the middle part, and the others are then opposite parts.

For the check formula, proceed as above with the circular parts corresponding to the three required elements.

Thus, if c and A are given, and a, b, and B are required:

To find a, consider the circular parts a, co. c, and co. A. Of these, a is the middle part, and co. c and co. A are opposite parts. Then, by Napier's rules,

$$\sin a = \cos(\text{co. } c)\cos(\text{co. } A) = \sin c \sin A$$

To find b, the circular parts are b, co. c, and co A; co. A is now the middle part, and b and co. c are adjacent parts. Then,

$$\sin(\text{co. } A) = \tan b \tan(\text{co.} c), \text{ or } \cos A = \tan b \cot c$$

To find B, the circular parts are co. B, co. c, and co. A; co. c is the middle part, and co. A and co. B are adjacent parts. Then,

$$\sin(\text{co. } c) = \tan(\text{co. } A)\tan(\text{co.} B), \text{ or } \cos c = \cot A \cot B$$

For the check formula, the circular parts are a, b, and co. B; a is the middle part, and b and co. B are adjacent parts. Then,

$$\sin a = \tan b \tan(\text{co. } B) = \tan b \cot B$$

EXAMPLES.

229. 1. Given $c = 70° 18'$, $A = 100°$. Find a, b, and B.

Using the rule of Art. 223, the three formulæ from Art. 218 are,

$$\sin A = \frac{\sin a}{\sin c}, \quad \cos A = \frac{\tan b}{\tan c}, \quad \cos c = \cot A \cot B.$$

That is,

$$\overset{+}{\sin a} = \overset{+}{\sin c}\,\overset{+}{\sin A},\quad \overset{-}{\tan b} = \overset{+}{\tan c}\,\overset{-}{\cos A},\quad \overset{-}{\cot B} = \overset{+}{\cos c}\,\overset{-}{\tan A}$$

Hence,

$$\begin{aligned}\log\sin a &= \log\sin c + \log\sin A\\ \log\tan b &= \log\tan c + \log\cos A\\ \log\cot B &= \log\cos c + \log\tan A\end{aligned}$$

Since $\tan b$ and $\cot B$ are negative, the *supplements* of the angles obtained from the tables must be taken, as explained in Art. 224. The element a is determined from its sine, but the ambiguity is removed by the principle of Art. 225; for, since A is greater than 90°, a must also be greater than 90°. Thus, in the case of a also, the supplement of the angle obtained from the tables must be taken.

When the supplement of an angle corresponding is to be taken, it may be conveniently expressed by writing 180° minus the element in the first member, as shown below in the cases of a, b, and B.

By the rule of Art. 227, the check formula is here

$$\tan B = \frac{\tan b}{\sin a}, \quad\text{or}\quad \sin a = \tan b \cot B.$$

The check formula should always be so written as to involve the functions used in determining the required parts. The values of $\log\tan b$ and $\log\cot B$ may then be taken from the first part of the work, and the sum should equal the previously found value of $\log\sin a$.

$$\begin{array}{rl}\log\sin c &= 9.973807-10\\ \log\sin A &= 9.993351-10\\ \hline \log\sin a &= 9.967158-10\\ 180^\circ - a &= 67^\circ\ 59'\ 50.6''\\ a &= 112^\circ\ 0'\ 9.4''\end{array}
\qquad
\begin{array}{rl}\log\tan c &= 0.446054\\ \log\cos A &= 9.239670-10\\ \hline \log\tan b &= 9.685724-10\\ 180^\circ - b &= 25^\circ\ 52'\ 20.9''\\ b &= 154^\circ\ 7'\ 39.1''\end{array}$$

$$\begin{array}{rl}\log\cos c &= 9.527753-10\\ \log\tan A &= 0.753681\\ \hline \log\cot B &= 0.281434\\ 180^\circ - B &= 27^\circ\ 36'\ 46.9''\\ B &= 152^\circ\ 23'\ 13.1''\end{array}$$

Check.

$$\begin{array}{rl}\log\tan b &= 9.685724-10\\ \log\cot B &= 0.281434\\ \hline \log\sin a &= 9.967158-10\end{array}$$

2. Given $a = 132° 6'$, $b = 77° 51'$. Find A, B, and c.

By Art. 223, the three formulæ are

$$\overset{-}{\tan A} = \frac{\overset{-}{\tan a}}{\underset{+}{\sin b}}, \quad \tan B = \frac{\tan b}{\sin a}, \quad \overset{-}{\cos c} = \overset{-}{\cos a}\, \overset{+}{\cos b}$$

Hence,

$$\begin{aligned} \log \tan A &= \log \tan a - \log \sin b \\ \log \tan B &= \log \tan b - \log \sin a \\ \log \cos c &= \log \cos a + \log \cos b \end{aligned}$$

By Art. 227, the check formula is

$$\cos c = \cot A \cot B, \text{ or } \cos c \tan A \tan B = 1$$

or, $$\log \cos c + \log \tan A + \log \tan B = \log 1 = 0$$

$$\begin{aligned} \log \tan a &= 0.044039 \\ \log \sin b &= \underline{9.990161} \\ \log \tan A &= 0.053878 \\ 180° - A &= 48° 32' 41.7'' \\ A &= 131° 27' 18.3'' \end{aligned}$$

$$\begin{aligned} \log \tan b &= 0.666967 \\ \log \sin a &= \underline{9.870390} \\ \log \tan B &= 0.796577 \\ B &= 80° 55' 26.6'' \end{aligned}$$

$$\begin{aligned} \log \cos a &= 9.826351 \\ \log \cos b &= \underline{9.323194} \\ \log \cos c &= 9.149545 \\ 180° - c &= 81° 53' 17.4'' \\ c &= 98° \;\; 6' 42.6'' \end{aligned}$$

Check.

$$\begin{aligned} \log \cos c &= 9.149545 \\ \log \tan A &= 0.053878 \\ \log \tan B &= \underline{0.796577} \\ \log 1 &= 0.000000 \end{aligned}$$

3. Given $B = 146° 15' 19''$, $a = 108° 3' 37''$. Find A, b, and c.

By Art 223, the three formulæ are

$$\sin B = \frac{\cos A}{\cos a}, \quad \tan B = \frac{\tan b}{\sin a}, \quad \cos B = \frac{\tan a}{\tan c}$$

That is,

$$\overset{-}{\cos A} = \overset{-}{\cos a}\, \overset{+}{\sin B}, \quad \overset{-}{\tan b} = \overset{+}{\sin a}\, \overset{-}{\tan B}, \quad \overset{+}{\tan c} = \frac{\overset{-}{\tan a}}{\underset{-}{\cos B}}$$

Hence,

$$\begin{aligned} \log \cos A &= \log \cos a + \log \sin B \\ \log \tan b &= \log \sin a + \log \tan B \\ \log \tan c &= \log \tan a - \log \cos B \end{aligned}$$

The check formula is $\cos A = \frac{\tan b}{\tan c}$.

That is, $$\log \cos A = \log \tan b - \log \tan c$$

$$\begin{array}{ll}
\log \cos a = 9.491386 & \log \sin a = 9.978057 \\
\log \sin B = \underline{9.744679} & \log \tan B = \underline{9.824806} \\
\log \cos A = 9.236065 & \log \tan b = 9.802863 \\
180^\circ - A = 80^\circ \ 5' \ 0.6'' & 180^\circ - b = 32^\circ 25' 15.3'' \\
A = 99^\circ 54' 59.4'' & b = 147^\circ 34' 44.7'' \\
\log \tan a = 0.486672 & \textit{Check.} \\
\log \cos B = \underline{9.919873} & \log \tan b = 9.802863 \\
\log \tan c = 0.566799 & \log \tan c = \underline{0.566799} \\
c = 74^\circ 49' 45.5'' & \log \cos A = 9.236064
\end{array}$$

Note. We observe here a difference of 1 in the sixth decimal place in the two values of $\log \cos A$. This does not necessarily indicate a mistake in the work, for such a difference might easily be due to the fact that the logarithms are only *approximately* correct to the sixth decimal place.

4. Given $A = 105^\circ 59'$, $a = 128^\circ 33'$; find b, B, and c.

By Art. 223,

$$\tan A = \frac{\tan a}{\sin b}, \quad \sin B = \frac{\cos A}{\cos a}, \quad \sin A = \frac{\sin a}{\sin c}$$

That is,

$$\overset{+}{\sin} b = \frac{\overset{-}{\tan} a}{\underset{-}{\tan A}}, \quad \overset{+}{\sin} B = \frac{\overset{-}{\cos} A}{\underset{-}{\cos a}}, \quad \sin c = \frac{\sin a}{\sin A}$$

Hence, $$\begin{aligned} \log \sin b &= \log \tan a - \log \tan A \\ \log \sin B &= \log \cos A - \log \cos a \\ \log \sin c &= \log \sin a - \log \sin A \end{aligned}$$

The check formula is $\sin B = \frac{\sin b}{\sin c}$.

That is, $$\log \sin B = \log \sin b - \log \sin c$$

All the required parts are here determined from their sines; and, as the ambiguity cannot be removed by the principles of

Arts. 225 or 226, both the angle obtained from the tables and its supplement must be retained in each case.

$$\begin{array}{rl}
\log\tan a &= 0.098617\\
\log\tan A &= \underline{0.542981}\\
\log\sin b &= 9.555636\\
b &= 21^\circ\ 3'\,58.7''\\
\text{or} & 158^\circ\,56'\ 1.3''
\end{array}
\qquad
\begin{array}{rl}
\log\cos A &= 9.439897\\
\log\cos a &= \underline{9.794626}\\
\log\sin B &= 9.645271\\
B &= 26^\circ\,13'\,18.3''\\
\text{or} & 153^\circ\,46'\,41.7''
\end{array}$$

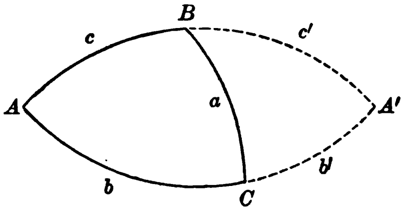
$$\begin{array}{rl}
\log\sin a &= 9.893243\\
\log\sin A &= \underline{9.982878}\\
\log\sin c &= 9.910365\\
c &= 54^\circ\,26'\,26.7''\\
\text{or} & 125^\circ\,33'\,33.3''
\end{array}
\qquad
\begin{array}{rl}
 & \textit{Check.}\\
\log\sin b &= 9.555636\\
\log\sin c &= \underline{9.910365}\\
\log\sin B &= 9.645271
\end{array}$$

It does not follow, however, that the parts can be combined promiscuously; for, by Art. 226, when c is less than 90°, a and b must be in the same quadrant, and when c is greater than 90°, a and b must be in different quadrants. Since a is given in the second quadrant, it follows that with the value of c less than 90° must be taken the value of b greater than 90°, and hence, also, by Art. 225, the value of B greater than 90°; and with the value of c greater than 90° must be taken the values of b and B less than 90°.

Thus, the two solutions are

$$c = 54^\circ\,26'\,26.7'',\quad b = 158^\circ\,56'\ 1.3'',\quad B = 153^\circ\,46'\,41.7'';$$

$$\text{or, } c = 125^\circ\,33'\,33.3'',\quad b = 21^\circ\ 3'\,58.7'',\quad B = 26^\circ\,13'\,18.3''.$$



The figure shows, in another way, why there should be two solutions in this case. For, if AB and AC be produced to A', forming the lune $ABA'C$, the triangle $A'BC$ has the side

a and the angle A' equal, respectively, to the side a and angle A of the triangle ABC, and both triangles are right-angled at C. And it is evident that the sides c' and b' and the angle CBA' are the supplements, respectively, of the sides c and b and the angle CBA.

Solve the following triangles:

5. Given $c = 137^\circ\ 23'$, $a = 160^\circ$.
6. Given $A = 50^\circ\ 21'$, $B = 122^\circ\ 40'$.
7. Given $a = 160^\circ$, $b = 38^\circ\ 28'$.
8. Given $B = 80^\circ\ 1'$, $b = 67^\circ\ 36'$.
9. Given $c = 81^\circ\ 53'$, $B = 112^\circ\ 3'$.
10. Given $a = 61^\circ\ 5'$, $B = 123^\circ\ 38'$.
11. Given $a = 61^\circ\ 39'$, $b = 144^\circ\ 12'$.
12. Given $A = 99^\circ\ 47'$, $a = 111^\circ\ 57'$
13. Given $c = 152^\circ\ 20'$, $b = 15^\circ\ 3'$.
14. Given $A = 62^\circ\ 59'$, $B = 37^\circ\ 4'$.
15. Given $A = 144^\circ\ 54'$, $a = 146^\circ\ 32'$.
16. Given $A = 144^\circ\ 54'$, $B = 101^\circ\ 14'$.
17. Given $a = 131^\circ\ 8'$, $B = 161^\circ\ 52'$.
18. Given $c = 114^\circ\ 32'$, $A = 73^\circ\ 7'$.
19. Given $a = 113^\circ\ 25'$, $b = 110^\circ\ 47'$.
20. Given $b = 137^\circ\ 9'$, $A = 74^\circ\ 51'$.
21. Given $c = 47^\circ\ 34'$, $A = 62^\circ\ 59'\ 10''$.
22. Given $b = 69^\circ\ 18'$, $c = 84^\circ\ 27'$.

SOLUTION OF QUADRANTAL TRIANGLES.

230. Since the polar triangle of a quadrantal triangle is a right triangle (Art. 187), we have only to solve this right triangle, and take the supplements of the parts found by the calculation.

Example. Given $c = 90°$, $a = 67° 38'$, $b = 48° 50'$. Find A, B, and C.

Denoting the polar right triangle by $A'B'C'$, we have, by Art. 186, (h),

$C' = 90°$, $A' = 112° 22'$, $B' = 131° 10'$; to find a', b', and c'.

By Art. 218,

$$\sin B' = \frac{\cos A'}{\cos a'},\quad \sin A' = \frac{\cos B'}{\cos b'},\quad \cos c' = \cot A' \cot B'$$

That is,

$$\overset{-}{\cos a'} = \frac{\overset{-}{\cos A'}}{\underset{+}{\sin B'}},\quad \overset{-}{\cos b'} = \frac{\overset{-}{\cos B'}}{\underset{+}{\sin A'}},\quad \overset{+}{\cos c'} = \overset{-}{\cot A'}\ \overset{-}{\cot B'}$$

or,

$$\log\cos a' = \log\cos A' - \log\sin B'$$
$$\log\cos b' = \log\cos B' - \log\sin A'$$
$$\log\cos c' = \log\cot A' + \log\cot B'$$

For the check,

$$\cos c' = \cos a' \cos b'$$

or,

$$\log\cos c' = \log\cos a' + \log\cos b'$$

$\log\cos A' = 9.580392$	$\log\cos B' = 9.818392$
$\log\sin B' = \underline{9.876678}$	$\log\sin A' = \underline{9.966033}$
$\log\cos a' = 9.703714$	$\log\cos b' = 9.852359$
$180° - a' = 59° 38' 9.6''$	$180° - b' = 44° 37' 6.2''$
$\log\cot A' = 9.614359$	*Check.*
$\log\cot B' = \underline{9.941714}$	$\log\cos a' = 9.703714$
$\log\cos c' = 9.556073$	$\log\cos b' = \underline{9.852359}$
$c' = 68° 54' 41.3''$	$\log\cos c' = 9.556073$

Thus, in the given quadrantal triangle,

$$A = 180° - a' = 59° 38' 9.6''$$
$$B = 180° - b' = 44° 37' 6.2''$$

and

$$C = 180° - c' = 111° 5' 18.7''$$

SOLUTION OF ISOSCELES TRIANGLES.

231. By Art. 186, (*d*), the perpendicular from the vertex to the base of an isosceles triangle bisects the base and the vertical angle, and divides it into two symmetrical right triangles. By solving one of these we can find the required parts of the given triangle.

Example. Given $a = 115°$, $b = 115°$, $C = 71° 48'$. Find A, B, and c.

Denoting the elements of one of the right triangles by A', B', C', a', b', and c', where C' is the right angle, we have

$$c' = a = 115°, \text{ and } A' = \tfrac{1}{2}C = 35° \, 54'$$

We are then to find the parts a' and B' in this triangle.

$$\sin A' = \frac{\sin a'}{\sin c'}, \quad \text{and} \quad \cos c' = \cot A' \cot B' \text{ (Art. 218)}$$

$$\text{or,} \quad \sin a' = \sin c' \sin A', \text{ and } \overset{-}{\cot B'} = \overset{-}{\cos c'} \, \overset{+}{\tan A'}$$

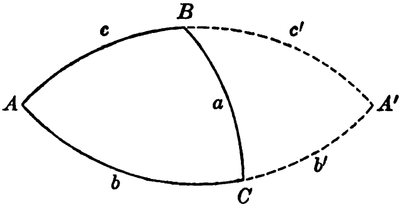
$$\begin{array}{ll}
\log \sin c' = 9.957276 & \log \cos c' = 9.625948 \\
\log \sin A' = \underline{9.768173} & \log \tan A' = \underline{9.859666} \\
\log \sin a' = 9.725449 & \log \cot B' = 9.485614 \\
\qquad a' = 32° \, 6' \, 8.7'' & 180° - B' = 72° \, 59' \, 23.5'' \\
 & \qquad B' = 107° \quad 0' \, 36.5''
\end{array}$$

Then, in the given isosceles triangle,

$$A = B = B' = 107° \, 0' \, 36.5'', \text{ and } c = 2a' = 64° \, 12' \, 17.4''$$

If the given parts are a, b, and A, and each is equal to 90°, the triangle is indeterminate, since any bi-rectangular triangle will satisfy the given conditions. The same is true if the given parts are A, B, and a, and each is equal to 90°.

232. If in a spherical oblique triangle the given parts are two sides or two angles which are supplements of each other, and one other element, the solution may be at once reduced to that of an isosceles triangle.



For, if a, b, and another element are given, and $a = 180° - b$, by completing the lune $ABA'C$ we form a triangle $A'BC$ in which the side $b' = 180° - b = a$; that is, the triangle $A'BC$ is isosceles.

Again, if A, B, and another element are given, and $A = 180° - B$, then, in the triangle $A'BC$, $A' = A$ and $A'BC = 180° - B = A$; that is, two angles of the triangle $A'BC$ are equal.

XVI. SOLUTION OF SPHERICAL OBLIQUE TRIANGLES.

233. In the solution of spherical oblique triangles we may distinguish six cases:

1. *Given a side and two adjacent angles.*
2. *Given two sides and their included angle.*
3. *Given the three sides.*
4. *Given the three angles.*
5. *Given two sides and the angle opposite to one of them.*
6. *Given two angles and the side opposite to one of them.*

234. By the application of the principles of Art. 186, (*h*), the solution of any example under Cases 2, 4, and 6, may be made to depend upon the solution of another example under Cases 1, 3, and 5, respectively, and *vice versa.* Thus it is not essential to consider more than three cases in the solution of spherical oblique triangles.

235. The student must bear carefully in mind the remarks made in regard to the solution of right triangles in Art. 224.

236. CASE I. *Given a side and two adjacent angles.*

Example. Given $A = 70°$, $B = 131° 18'$, $c = 116°$. Find a, b, and C.

By Napier's Analogies (Arts. 203, 204), we have

$$\frac{\sin\frac{1}{2}(B+A)}{\sin\frac{1}{2}(B-A)} = \frac{\tan\frac{1}{2}c}{\tan\frac{1}{2}(b-a)}$$

$$\frac{\cos\frac{1}{2}(B+A)}{\cos\frac{1}{2}(B-A)} = \frac{\tan\frac{1}{2}c}{\tan\frac{1}{2}(b+a)}$$

Whence, $$\tan\tfrac{1}{2}(b-a) = \frac{\sin\frac{1}{2}(B-A)}{\sin\frac{1}{2}(B+A)}\tan\tfrac{1}{2}c$$

$$\tan\tfrac{1}{2}(b+a) = \frac{\cos\frac{1}{2}(B-A)}{\cos\frac{1}{2}(B+A)}\tan\tfrac{1}{2}c$$

or, $$\tan\tfrac{1}{2}(b-a) = \sin\tfrac{1}{2}(B-A)\operatorname{cosec}\tfrac{1}{2}(B+A)\tan\tfrac{1}{2}c$$

$$\overset{-}{\tan\tfrac{1}{2}(b+a)} = \overset{+}{\cos\tfrac{1}{2}(B-A)}\sec\tfrac{1}{2}\overset{-}{(B+A)}\,\overset{+}{\tan\tfrac{1}{2}c}$$

From the data,

$$\tfrac{1}{2}(B-A) = 30° 39', \quad \tfrac{1}{2}(B+A) = 100° 39', \quad \tfrac{1}{2}c = 58°$$

$$\begin{array}{rlrl}
\log\sin\frac{1}{2}(B-A) &= 9.707393 & \log\cos\frac{1}{2}(B-A) &= 9.934649 \\
\log\operatorname{cosec}\frac{1}{2}(B+A) &= 0.007546 & \log\sec\frac{1}{2}(B+A) &= 0.733277 \\
\log\tan\frac{1}{2}c &= \underline{0.204211} & \log\tan\frac{1}{2}c &= \underline{0.204211} \\
\log\tan\frac{1}{2}(b-a) &= 9.919150 & \log\tan\frac{1}{2}(b+a) &\overset{n}{=} 0.872137 \\
\therefore\ \frac{1}{2}(b-a) &= 39° 41' 50.5'' & 180° - \frac{1}{2}(b+a) &= 82° 21' 16.7'' \\
 & & \therefore\ \frac{1}{2}(b+a) &= 97° 38' 43.3''
\end{array}$$

Then, $$a = \tfrac{1}{2}(b+a) - \tfrac{1}{2}(b-a) = \ 57° 56' 52.8''$$

$$b = \tfrac{1}{2}(b+a) + \tfrac{1}{2}(b-a) = 137° 20' 33.8''$$

To find C, we have, by Art. 205,

$$\cot\tfrac{1}{2}C=\frac{\sin\frac{1}{2}(b+a)}{\sin\frac{1}{2}(b-a)}\tan\tfrac{1}{2}(B-A)$$

$$=\sin\tfrac{1}{2}(b+a)\operatorname{cosec}\tfrac{1}{2}(b-a)\tan\tfrac{1}{2}(B-A)$$

$$\begin{aligned}\log\sin\tfrac{1}{2}(b+a)&=9.996122\\ \log\operatorname{cosec}\tfrac{1}{2}(b-a)&=0.194681\\ \log\tan\tfrac{1}{2}(B-A)&=\underline{9.772745}\\ \log\cot\tfrac{1}{2}C&=9.963548\\ \tfrac{1}{2}C&=47^\circ\,24'\ \ 6.1''\\ \therefore\ C&=94^\circ\,48'\,12.2''\end{aligned}$$

The value of C may also be determined by the formula (Art. 205),

$$\cot\tfrac{1}{2}C=\frac{\cos\frac{1}{2}(b+a)}{\cos\frac{1}{2}(b-a)}\tan\tfrac{1}{2}(B+A)$$

The triangle is always possible for any values of the given elements.

237. Case II. *Given two sides and their included angle.*

Example. Given $b=137^\circ\,20'$, $c=116^\circ$, $A=70^\circ$. Find B, C, and a.

By Napier's Analogies (Art. 205), we have

$$\frac{\sin\frac{1}{2}(b+c)}{\sin\frac{1}{2}(b-c)}=\frac{\cot\frac{1}{2}A}{\tan\frac{1}{2}(B-C)},\quad\frac{\cos\frac{1}{2}(b+c)}{\cos\frac{1}{2}(b-c)}=\frac{\cot\frac{1}{2}A}{\tan\frac{1}{2}(B+C)}$$

Whence, $$\tan\tfrac{1}{2}(B-C)=\frac{\sin\frac{1}{2}(b-c)}{\sin\frac{1}{2}(b+c)}\cot\tfrac{1}{2}A$$

$$\tan\tfrac{1}{2}(B+C)=\frac{\cos\frac{1}{2}(b-c)}{\cos\frac{1}{2}(b+c)}\cot\tfrac{1}{2}A$$

or, $$\begin{aligned}\underset{-}{\tan\tfrac{1}{2}(B-C)}&=\underset{+}{\sin\tfrac{1}{2}(b-c)}\,\underset{-}{\operatorname{cosec}\tfrac{1}{2}(b+c)}\,\underset{+}{\cot\tfrac{1}{2}A}\\ \tan\tfrac{1}{2}(B+C)&=\cos\tfrac{1}{2}(b-c)\sec\tfrac{1}{2}(b+c)\cot\tfrac{1}{2}A\end{aligned}$$

From the data,

$$\tfrac{1}{2}(b-c)=10^\circ\,40',\quad\tfrac{1}{2}(b+c)=126^\circ\,40',\quad\tfrac{1}{2}A=35^\circ$$

From which we find,

$$\tfrac{1}{2}(B-C)=18^\circ\,14'\,23.4'', \quad \tfrac{1}{2}(B+C)=113^\circ\,2'\,56.9''$$

Then,
$$B=\tfrac{1}{2}(B+C)+\tfrac{1}{2}(B-C)=131^\circ\,17'\,20.3''$$
$$C=\tfrac{1}{2}(B+C)-\tfrac{1}{2}(B-C)=\ \ 94^\circ\,48'\,33.5''$$

To find a, we have, by Art. 203,

$$\tan\tfrac{1}{2}a=\frac{\sin\tfrac{1}{2}(B+C)}{\sin\tfrac{1}{2}(B-C)}\tan\tfrac{1}{2}(b-c)$$
$$=\sin\tfrac{1}{2}(B+C)\operatorname{cosec}\tfrac{1}{2}(B-C)\tan\tfrac{1}{2}(b-c)$$

From which we obtain

$$a=57^\circ\,56'\,55.6''$$

The value of a may also be determined by the formula (Art. 204),

$$\tan\tfrac{1}{2}a=\frac{\cos\tfrac{1}{2}(B+C)}{\cos\tfrac{1}{2}(B-C)}\tan\tfrac{1}{2}(b+c)$$

The triangle is always possible for any values of the given elements.

238. Case III. *Given the three sides.*

Example. Given $a=57^\circ\,57'$, $b=137^\circ\,21'$, $c=116^\circ$. Find A, B, and C.

By Art. 198,

$$\tan\tfrac{1}{2}A=\sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s\sin(\ \ -a)}}$$

$$\tan\tfrac{1}{2}B=\sqrt{\frac{\sin(s-c)\sin(s-a)}{\sin s\sin(s-b)}}$$

$$\tan\tfrac{1}{2}C=\sqrt{\frac{\sin(s-a)\sin(s-b)}{\sin s\sin(s-c)}}$$

From the data,

$$2s=a+b+c=311^\circ\,18', \text{ or } s=155^\circ\,39'$$

Then,

$$s-a=97^\circ\,42', \quad s-b=18^\circ\,18', \quad s-c=39^\circ\,39'$$

$$\begin{aligned}\log\sin(s-b) &= 9.496919\\ \log\sin(s-c) &= 9.804886\\ \log\operatorname{cosec}(s-a) &= 0.003934\\ \log\operatorname{cosec} s &= 0.384777\\ & 2\overline{)9.690516}\\ \log\tan\tfrac{1}{2}A &= 9.845258\\ \tfrac{1}{2}A &= 35^\circ\,0'\ \ 6.9''\\ \therefore\ A &= 70^\circ\,0'\,13.8''\end{aligned}$$

Similarly, we find

$$B = 131^\circ\,18'\,31.8'',\ \ C = 94^\circ\,48'\,5.6''$$

The following forms of the equations of Art. 198 will be found convenient when all the angles are required. (Compare Art. 162.)

If $k = \sqrt{\dfrac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}}$

then,

$$\tan\tfrac{1}{2}A = \frac{k}{\sin(s-a)}$$

$$\tan\tfrac{1}{2}B = \frac{k}{\sin(s-b)}$$

$$\tan\tfrac{1}{2}C = \frac{k}{\sin(s-c)}$$

The values of A, B, and C may also be found by aid of the formulæ of Arts. 196 or 197. If all the angles are required, the formulæ of Art. 198 are the most convenient, as only four different logarithms are required. If but one angle is required, the formulæ of Art. 197 are to be preferred, as they involve the least work.

The triangle is always possible for any values of the given elements which satisfy the conditions of Art. 186, (a) and (e); that is, if $a + b + c < 360^\circ$, and no side is greater than the sum of the other two.

239. Case IV. *Given the three angles.*

Example. Given $A = 70°$, $B = 131° \ 18'$, $C = 94° \ 48'$. Find a, b, and c.

By Art. 201,

$$\tan \tfrac{1}{2} a = \sqrt{- \frac{\cos S \cos (S - A)}{\cos (S - B) \cos (S - C)}}$$

$$\tan \tfrac{1}{2} b = \sqrt{- \frac{\cos S \cos (S - B)}{\cos (S - C) \cos (S - A)}}$$

$$\tan \tfrac{1}{2} c = \sqrt{- \frac{\cos S \cos (S - C)}{\cos (S - A) \cos (S - B)}}$$

Note. Since it has been proved in Art. 202 that the values of the tangents of the half sides are always real, we need pay no attention to the negative signs in the formulæ.

From the data,

$$2S = A + B + C = 296° \ 6', \text{ or } S = 148° \ 3'$$

Then,

$$S - A = 78° \ 3', \ S - B = 16° \ 45', \ S - C = 53° \ 15'$$

$$\begin{aligned}
\log \cos S &= 9.928657 \\
\log \cos (S - A) &= 9.316092 \\
\log \sec (S - B) &= 0.018829 \\
\log \sec (S - C) &= \underline{0.223063} \\
& \quad 2 \overline{)9.486641} \\
\log \tan \tfrac{1}{2} a &= 9.743320 \\
\tfrac{1}{2} a &= 28° \ 58' \ 33'' \\
\therefore a &= 57° \ 57' \ 6''
\end{aligned}$$

In a similar manner, we find

$$b = 137° \ 20' \ 25.8'', \quad c = 115° \ 59' \ 40.4''$$

If all the sides are required, the following modifications of the formulæ of Art. 201 may be used:

$$\text{If } K = \sqrt{\frac{- \cos S}{\cos (S - A) \cos (S - B) \cos (S - C)}}$$

then,
$$\tan\tfrac{1}{2}a = K\cos(S-A)$$
$$\tan\tfrac{1}{2}b = K\cos(S-B)$$
$$\tan\tfrac{1}{2}c = K\cos(S-C)$$

The values of a, b, and c may also be found by aid of the formulæ of Arts. 199 or 200. If all the sides are required, the formulæ of Art. 201 are the most convenient, as only four different logarithms are required. If but one side is required, the formulæ of Art. 200 are to be preferred, as they involve the least work.

The triangle is always possible for any values of the given elements which satisfy the condition of Art. 186, (*f*), and those shown to be true in Art. 202. That is, $A+B+C$ must be $>180°$ and $<540°$, and each of the quantities $B+C-A$, $C+A-B$, and $A+B-C$, must lie between $180°$ and $-180°$.

240. Case V. *Given two sides and the angle opposite to one of them.*

Example 1. Given $a = 57° 57'$, $b = 137° 21'$, $B = 131° 18'$. Find A, C, and c.

By Art 192, $\frac{\sin A}{\sin B} = \frac{\sin a}{\sin b}$, or $\sin A = \frac{\sin a \sin B}{\sin b}$

That is, $\sin A = \sin a \operatorname{cosec} b \sin B$

$$\begin{aligned}
\log\sin a &= 9.928183\\
\log\operatorname{cosec} b &= 0.169079\\
\log\sin B &= \underline{9.875793}\\
\log\sin A &= 9.973055
\end{aligned}$$

$$\therefore A = 70° \, 1' \, 30.3'', \text{ or } 109° \, 58' \, 29.7''$$

To find C and c, we have, by Arts. 203 and 205,

$$\left.\begin{aligned}
\cot\tfrac{1}{2}C &= \frac{\sin\frac{1}{2}(b+a)}{\sin\frac{1}{2}(b-a)}\tan\tfrac{1}{2}(B-A)\\
\tan\tfrac{1}{2}c &= \frac{\sin\frac{1}{2}(B+A)}{\sin\frac{1}{2}(B-A)}\tan\tfrac{1}{2}(b-a)
\end{aligned}\right\} \quad \text{(A)}$$

$$\text{Or,}\quad \cot\tfrac{1}{2}C = \sin\tfrac{1}{2}(b+a)\operatorname{cosec}\tfrac{1}{2}(b-a)\tan\tfrac{1}{2}(B-A)$$
$$\tan\tfrac{1}{2}c = \sin\tfrac{1}{2}(B+A)\operatorname{cosec}\tfrac{1}{2}(B-A)\tan\tfrac{1}{2}(b-a)$$

Using the first value of A, we have

$$B+A = 201^\circ\, 19'\, 30.3'', \qquad B-A = 61^\circ\, 16'\, 29.7''$$
$$\text{or,}\quad \tfrac{1}{2}(B+A) = 100^\circ\, 39'\, 45.15'', \quad \tfrac{1}{2}(B-A) = 30^\circ\, 38'\, 14.85''$$

Also,

$$\tfrac{1}{2}(b+a) = 97^\circ\, 39', \qquad \tfrac{1}{2}(b-a) = 39^\circ\, 42'$$

From which we obtain

$$C = 94^\circ\, 50'\, 8.8'', \quad c = 116^\circ\, 1'\, 18.2''$$

Using the second value of A, we have

$$B+A = 241^\circ\, 16'\, 29.7'', \qquad B-A = 21^\circ\, 19'\, 30.3''$$
$$\text{or,}\quad \tfrac{1}{2}(B+A) = 120^\circ\, 38'\, 14.85'', \quad \tfrac{1}{2}(B-A) = 10^\circ\, 39'\, 45.15''$$

From which we have

$$C = 147^\circ\, 25'\, 53'', \quad c = 150^\circ\, 57'\, 24.4''$$

Thus, the two solutions are

$$A = 70^\circ\, 1'\, 30.3'', \quad C = 94^\circ\, 50'\, 8.8'', \quad c = 116^\circ\, 1'\, 18.2''$$
$$\text{or,}\quad A = 109^\circ\, 58'\, 29.7'', \quad C = 147^\circ\, 25'\, 53'', \quad c = 150^\circ\, 57'\, 24.4''$$

241. The values of $\cot\frac{1}{2}C$ and $\tan\frac{1}{2}c$ given in the equations (**A**) of the preceding article must be positive, since $\frac{1}{2}C$ and $\frac{1}{2}c$ are less than 90°. Hence, if either of the values of A is such as to make $\cot\frac{1}{2}C$ and $\tan\frac{1}{2}c$ negative, there will be no triangle corresponding. Thus there will sometimes be only one solution, and sometimes no solution in examples under Case V.

After the values of A have been obtained, the number of solutions may be readily determined by inspection. For, since $\frac{1}{2}(b+a)$ and $\frac{1}{2}(B+A)$ are less than 180°, $\sin\frac{1}{2}(b+a)$ and $\sin\frac{1}{2}(B+A)$ are positive. Hence, if $B-A$ and $b-a$ have the *same* sign, $\cot\frac{1}{2}C$ and $\tan\frac{1}{2}c$ are both positive, and the triangle is possible; but if they have *different* signs, $\cot\frac{1}{2}C$ and $\tan\frac{1}{2}c$ are both negative, and there is no triangle corresponding.

That is, *only those values of A can be retained which are greater or less than B according as a is greater or less than b.*

To illustrate, in the example of Art. 240, $a < b$; and, since both values of A are less than B, we have two solutions.

242. *Example* 2. Given $a = 57° 56'$, $c = 116°$, $C = 94° 48'$. Find A, B, and b.

Here, $$\frac{\sin A}{\sin C} = \frac{\sin a}{\sin c}$$

or, $$\sin A = \frac{\sin a \sin C}{\sin c} = \sin a \operatorname{cosec} c \sin C$$

$$\begin{aligned} \log \sin a &= 9.928104 \\ \log \operatorname{cosec} c &= 0.046340 \\ \log \sin C &= \underline{9.998474} \\ \log \sin A &= 9.972918 \end{aligned}$$

$$\therefore A = 69° 58' 31.2'', \text{ or } 110° 1' 28.8''$$

Since $a < c$, only values of A less than C can be retained; hence we have but one solution, which corresponds to the first value of A.

To find B and b, we have, by Arts. 203 and 205,

$$\cot \tfrac{1}{2} B = \frac{\sin \frac{1}{2}(c + a)}{\sin \frac{1}{2}(c - a)} \tan \tfrac{1}{2}(C - A)$$

$$\tan \tfrac{1}{2} b = \frac{\sin \frac{1}{2}(C + A)}{\sin \frac{1}{2}(C - A)} \tan \tfrac{1}{2}(c - a)$$

Using the first value of A, we have

$$C + A = 164° 46' 31.2'', \qquad C - A = 24° 49' 28.8''$$

or, $$\tfrac{1}{2}(C + A) = 82° 23' 15.6'', \qquad \tfrac{1}{2}(C - A) = 12° 24' 44.4''$$

Also,

$$\tfrac{1}{2}(c + a) = 86° 58', \qquad \tfrac{1}{2}(c - a) = 29° 2'$$

From which we obtain

$$B = 131° 16' 20'', \quad b = 137° 19' 13.8''$$

Thus, the only solution is

$$A = 69° 58' 31.2'', \quad B = 131° 16' 20'', \quad b = 137° 19' 13.8''$$

Example 3. Given $b = 126°$, $c = 70°$, $B = 56°$. Find a, A, and C.

Here, $$\frac{\sin C}{\sin B} = \frac{\sin c}{\sin b}, \text{ or } \sin C = \frac{\sin c \sin B}{\sin b}$$

$$\begin{aligned} \log \sin c &= 9.972986 \\ \log \operatorname{cosec} b &= 0.092042 \\ \log \sin B &= \underline{9.918574} \\ \log \sin C &= 9.983602 \\ \therefore C &= 74° 21' 13.6'', \text{ or } 105° 38' 46.4'' \end{aligned}$$

Since both values of C are greater than B, while c is less than b, it follows that there is no solution.

243. If the given parts are such as to make the log sine of the first angle obtained in the computation positive, the triangle will of course be impossible; in this case also there will be no solution.

244. Rules have been given by some authors for determining by inspection of the given elements when there are two solutions, when only one, and when none. These rules are very long and inconvenient of application, and it has not been thought worth while to give them here. In practice it is preferable to determine the number of solutions by the method of Art. 241.

245. CASE VI. *Given two angles and the side opposite to one of them.*

Example. Given $A = 109° 58'$, $B = 131° 18'$, $b = 137° 20'$. Find a, c, and C.

Here, $$\frac{\sin a}{\sin b} = \frac{\sin A}{\sin B}, \text{ or } \sin a = \frac{\sin b \sin A}{\sin B}$$

$$\begin{aligned} \log \sin b &= 9.831058 \\ \log \sin A &= 9.973078 \\ \log \operatorname{cosec} B &= \underline{0.124207} \\ \log \sin a &= 9.928343 \\ \therefore a &= 57° 59' 0.8'', \text{ or } 122° 0' 59.2'' \end{aligned}$$

To find C and c, we have, by Arts. 203 and 205,

$$\left.\begin{aligned}\cot\tfrac{1}{2}C &= \frac{\sin\frac{1}{2}(b+a)}{\sin\frac{1}{2}(b-a)}\tan\tfrac{1}{2}(B-A)\\ \tan\tfrac{1}{2}c &= \frac{\sin\frac{1}{2}(B+A)}{\sin\frac{1}{2}(B-A)}\tan\tfrac{1}{2}(b-a)\end{aligned}\right\}\qquad(\text{A})$$

Using the first value of a, we have

$$b+a=195^\circ\,19'\,0.8'', \qquad b-a=79^\circ\,20'\,59.2''$$

or, $\frac{1}{2}(b+a)=97^\circ\,39'\,30.4''$, $\frac{1}{2}(b-a)=39^\circ\,40'\,29.6''$

Also,

$$\tfrac{1}{2}(B+A)=120^\circ\,38', \qquad \tfrac{1}{2}(B-A)=10^\circ\,40'$$

From which we find

$$C=147^\circ\,24'\,12.8'', \quad c=150^\circ\,55'\,21''$$

Using the second value of a, we have

$$b+a=259^\circ\,20'\,59.2'', \qquad b-a=15^\circ\,19'\,0.8''$$

or, $\frac{1}{2}(b+a)=129^\circ\,40'\,29.6''$, $\frac{1}{2}(b-a)=7^\circ\,39'\,30.4''$

From which we obtain

$$C=85^\circ\,11'\,1.4'', \quad c=64^\circ\,1'\,5.8''$$

Thus, the two solutions are

$a=57^\circ\,59'\,0.8''$, $C=147^\circ\,24'\,12.8''$, $c=150^\circ\,55'\,21''$

or, $a=122^\circ\,0'\,59.2''$, $C=85^\circ\,11'\,1.4''$, $c=64^\circ\,1'\,5.8''$

246. In Case VI., as in Case V., there are sometimes two solutions, sometimes only one, and sometimes none; and it may be shown by aid of the equations (A) of the preceding article, exactly as in Art. 241, that *only those values of a can be retained which are greater or less than b according as A is greater or less than B.*

Also, if the data are such as to make $\log\sin a$ positive, there will be no solution corresponding.

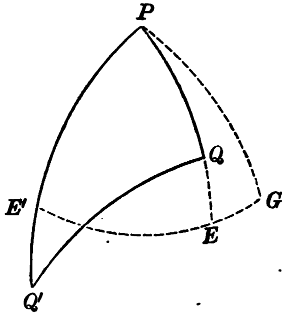
Rules have been given in this case, as well as in Case V., for determining by inspection when there are two solutions, when only one, and when none; but in practice the method just stated is preferable.

EXAMPLES.

247. Solve the following triangles:

1. Given $A = 50^\circ 10' 4''$, $B = 135^\circ 5' 34''$, $c = 70^\circ 20' 18''$.
2. Given $a = 101^\circ$, $b = 49^\circ$, $c = 60^\circ$.
3. Given $A = 119^\circ 58' 40''$, $B = 130^\circ 3' 10''$, $C = 79^\circ 46' 30''$.
4. Given $b = 99^\circ 41'$, $c = 64^\circ 23'$, $B = 95^\circ 38'$.
5. Given $a = 60^\circ 35' 40''$, $c = 98^\circ 2' 20''$, $B = 10^\circ 16' 40''$.
6. Given $A = 132^\circ 16'$, $B = 139^\circ 44'$, $b = 127^\circ 30'$.
7. Given $B = 66^\circ 25' 23''$, $A = 133^\circ 48' 17.2''$, $a = 81^\circ 12' 56''$.
8. Given $a = 40^\circ 6'$, $b = 118^\circ 22'$, $A = 29^\circ 43'$.
9. Given $a = 61^\circ$, $b = 39^\circ$, $c = 92^\circ$.
10. Given $a = 70^\circ$, $b = 120^\circ$, $C = 50^\circ$.
11. Given $B = 115^\circ 37'$, $C = 80^\circ 19'$, $c = 84^\circ 22'$.
12. Given $a = 115^\circ 18' 46''$, $c = 146^\circ 21' 24''$, $C = 141^\circ 11' 37''$.
13. Given $A = 50^\circ$, $C = 135^\circ$, $b = 70^\circ$.
14. Given $a = 109^\circ 18' 3''$, $c = 82^\circ 0' 15.3''$, $A = 107^\circ 37' 28''$.
15. Given $A = 91^\circ$, $B = 85^\circ$, $C = 72^\circ$.
16. Given $b = 122^\circ 36'$, $C = 81^\circ 13'$, $B = 72^\circ 55'$.
17. Given $A = 62^\circ 32' 47.26''$, $C = 102^\circ 8' 21.8''$, $a = 65^\circ 11' 13''$.
18. Given $B = 31^\circ 42'$, $C = 122^\circ 13' 40''$, $a = 40^\circ 31' 20''$.
19. Given $A = 138^\circ 15' 50''$, $B = 31^\circ 11' 10''$, $C = 35^\circ 50'$.
20. Given $c = 40^\circ 5' 30''$, $b = 118^\circ 22' 10''$, $C = 29^\circ 42' 40''$.
21. Given $b = 120^\circ 30'$, $c = 70^\circ 20'$, $A = 50^\circ 10'$.
22. Given $a = 162^\circ 23'$, $b = 15^\circ 37'$, $B = 125^\circ 2'$.
23. Given $A = 61^\circ 37' 50''$, $C = 139^\circ 54' 30''$, $c = 150^\circ 17' 30''$.
24. Given $a = 62^\circ$, $b = 54^\circ$, $c = 98^\circ$.

In questions concerning geodesy or navigation, we may regard the earth as a sphere. The shortest path between any two points is then the arc of a great circle which joins them; and the angles between this arc and the meridians of the points determine the bearings of the points from each other.



Thus, if Q and Q' are the points and PQ and PQ' their meridians, the angle PQQ' determines the bearing of Q' from Q, and the angle $PQ'Q$ determines the bearing of Q from Q'.

If the latitudes and longitudes of Q and Q' are given, it is possible to determine the arc QQ' and the angles PQQ' and $PQ'Q$ by the solution of a spherical triangle.

For, if EE' is the equator, and PG the meridian of Greenwich, the angle $QPQ' = Q'PG - QPG =$ longitude of Q' − longitude of Q. Also, $PQ = PE - QE = 90° -$ latitude of Q, and $PQ' = PE' + E'Q' = 90° +$ latitude of Q'. Thus two sides and the included angle of the triangle PQQ' are known, and the remaining elements may be calculated.

Note. When the arc QQ' has been found in angular measure, its length in miles may be calculated by the method explained in Art. 181; in the following examples the diameter of the earth is taken as 7912 miles.

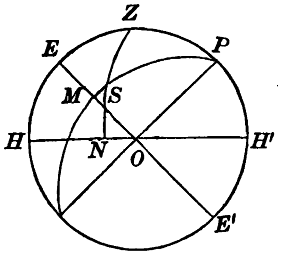
25. Boston lies in lat. 42° 21′ 22″ N., lon. 71° 4′ 9″ W.; and the latitude of Greenwich is 51° 28′ 38″ N. Find the shortest distance in miles between the places, and the bearing of each place from the other.

26. Calcutta lies in lat. 22° 33′ 5″ N., lon. 88° 19′ 2″ E.; and Valparaiso lies in lat. 33° 1′ 56″ S., lon. 71° 41′ 45″ W.

Find the shortest distance in miles between the places, and the bearing of each place from the other.

27. Sandy Hook lies in lat. 40° 27′ 35″ N., lon. 74° 0′ 48″ W.; and Queenstown lies in lat. 51° 50′ 4″ N., lon. 8° 19′ W. In what latitude does a great circle course from Sandy Hook to Queenstown cross the meridian of 50° W.?

If the latitude of a place is known, and the altitude and declination of the sun, the solution of a spherical triangle serves to determine the hour of the day at the time and place of observation.



Thus, let O be the observer's position; P the celestial north pole; EE' the celestial equator; HH' the horizon; Z the zenith; S the sun's position; PSM a meridian passing through the sun's place; and ZSN a great circle passing through Z and S.

Then SM is the sun's declination, SN its altitude, and EZ the latitude of the place of observation.

Then, in the triangle SPZ, the side $SP = PM - SM = 90° -$ the sun's declination; $SZ = ZN - SN = 90° -$ the sun's altitude; and $PZ = EP - EZ = 90° -$ the latitude of the place. That is, the three sides of the triangle SPZ are known, and the angle SPZ may be calculated.

If 24 hours be multiplied by the ratio of this angle to 360°, we have the time required for the sun to move from S to the meridian EP; hence, if this time be subtracted from 12 o'clock, if the observation is taken in the morning, or added, if taken in the afternoon, we obtain the hour of the day at the time and place of observation.

If the Greenwich time of the observation is noted on a chronometer, the difference between this and the local time as calculated above determines the longitude of the place. In reducing time to longitude, it should be borne in mind that 24 hours of time correspond to 360° of longitude; that is, one hour of time corresponds to 15° of longitude, one minute to 15′, and one second to 15″.

28. A mariner observes the sun's altitude to be 14° 18′, its declination being 18° 36′ N. If the latitude of the vessel is 50° 13′ N., and the observation is made in the morning, find the hour of the day. If the observation is taken at 9 A.M. Greenwich time, what is the longitude of the vessel?

29. What will be the altitude of the sun at 4 o'clock P.M. in San Francisco, lat. 37° 47′ 35″ N., its declination being 12° S.?

30. In Melbourne, lat. 37° 48′ 36″ S., the altitude of the sun is observed to be 25° 46′. If the sun's declination is 3° S., and the observation is made in the morning, find the hour of the day.

31. At what hour will the sun rise in Boston, lat. 42° 21′ 22″ N., when its declination is 15° N.?

Note. At sunrise the sun's altitude is 0, so that the arc SZ becomes 90°.

ANSWERS TO THE EXAMPLES.

Art. 7; page 4.

9. $28° \, 38' \, 52.4''$.

13. $42° \, 58' \, 18.6''$.

14. $114° \, 35' \, 29.6''$.

15. $100° \, 54' \, 5.1''$.

16. $30° \, 40' \, 33.8''$.

Art. 52; pages 40, 41.

9. $2n\pi + \frac{3\pi}{4}$.

10. $2n\pi + \frac{4\pi}{3}$.

11. $2n\pi + \frac{\pi}{3}$.

12. $2n\pi - \frac{\pi}{4}$.

Art. 83; page 68.

3. $n\pi$ or $2n\pi \pm \frac{\pi}{3}$.

4. $n\pi$ or $n\pi \pm \frac{\pi}{4}$.

5. $\sin^{-1}\left(\frac{\sqrt{5}-1}{2}\right)$.

6. $n\pi + \frac{\pi}{2}$ or $n\pi \pm \frac{\pi}{4}$.

7. $\sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right)$.

8. $n\pi \pm \frac{\pi}{6}$.

9. $n\pi + \frac{\pi}{4}$.

10. $\tan^{-1}\left(\frac{\sin a \sin b}{1 - \sin a \cos b}\right)$.

Art. 95; page 73.

1. 1.681241.

2. 2.644438.

3. 1.748188.

4. 1.991226.

5. 1.924279.

6. 2.753582.

7. 2.225309.

8. 3.848558.

9. 2.702430.

10. 3.489536.

11. 4.191785.

12. 4.158543.

Art. 96; page 74.

1. 1.176091.
2. 2.096910.
3. 0.154902.
4. 2.243038.
5. 0.522879.
6. 1.045758.
7. 0.853872.
8. 1.066947.
9. 0.735954.

Art. 97; page 74.

1. 0.286273.
2. 3.612360.
3. 4.604095−10.
4. 9.893973−10.
5. 0.878494.
6. 4.812327−10.

Art. 98; page 75.

1. 0.068160.
2. 0.460070.
3. 0.351447.
4. 0.937530.
5. 1.017051.
6. 1.056811.

Art. 104; page 80.

5. 0.011739.
6. 1.527511.
7. 8.780210−10.
8. 4.812917.
9. 7.013150−10.
10. 2.960116.
11. 9.942550−10.
12. 3.863506.
13. 8.640409 −10.
14. 4.89381.
15. 1.718451.
16. 7.4984240−10.
17. 9.275374−10.
18. 1.9792784.

Art. 105; pages 81, 82.

4. 6.61005.
5. 55606.5.
6. .011089.
7. 186.334.
8. .223905.
9. 1000.06.
10. 9.77667.
11. .00130514.
12. .034277.
13. 46.7929.
14. 11.327.
15. 8.63076.
16. .2070207.
17. .00548803.
18. 734.9114.

Art. 109; pages 87, 88.

1. .0341657.
2. .650573.
3. 13560.2.
4. .136085.
5. 1.14720.
6. 1.41421.
7. 1.49535.
8. .0655264.
9. – 1.97221.
10. 458.623.
11. –.000113607.
12. 5.88336.
13. 1.19491.
14. .788544. 7
15. .680192.
16. 2.24328.
17. .158489.
18. –.191680.
19. .644349.
20. .501126.
21. 1.09872.
22. 1.06179. 8
23. 1.09328.
24. .703621.
25. .580799.
26. –.631188.
27. 24.6196.
28. .297812.
29. 98.4295.
30. 1.65900.
31. 3.07616.
32. .867674.
33. – 2.09389.
34. .0597255.
35. .588142.
36. 1.80446.
37. .00323011.
38. .0334343.

Art. 117; pages 92, 93.

1. 7.
3. – 6.
5. 7.
7. 6.
9. 1.56937.
10. 2.44958.
11. 2.00906.
12. 3.96913.
15. – 1.07009.
13. 7.18923.
14. – 2.4578.

Art. 128; page 100.

3. 9.911700 – 10.
4. 9.945804 – 10.
5. 0.651899.
6. 9.999958 – 10.
7. 0.852277.
8. 9.914398 – 10.
9. 9.937383 – 10.
10. 9.954213 – 10.
11. 0.963531.
12. 3.07885.
13. .832806.
14. .75301.

Art. 129; pages 102, 103.

3. 83° 5′ 13.7″.
4. 33° 0′ 16.2″.
5. 46° 50′ 33″.
6. 31° 20′ 28.4″.
7. 80° 16′ 9″.
8. 8° 53′ 30.2″.
10. 5° 33′ 43.2″.
11. 26° 33′ 54.1″.
12. 33° 41′ 24.3″.
13. 41° 48′ 37.1″.
14. 61° 52′ 28.3″.
15. 5° 42′ 38.2″.

Art. 130; page 103.

2. 0.085603.

3. 0.765716.

4. $66° 33' 29.1''$.

5. $40° 21' 2.8''$.

Art. 133; page 106.

1. $6.849928 - 10$.

2. $7.031536 - 10$.

3. 2.911304.

4. $7.063608 - 10$.

5. $3' 11.706''$.

6. $5' 5.6562''$.

7. $89° 50' 47.4768''$.

Art. 140; pages 110, 111.

1. $a = 264.952$, $b = 75.9488$.
2. $A = 51° 52' 9.7''$, $B = 38° 7' 50.3''$, $c = 811.715$.
3. $c = 14.3056$, $a = 5.41295$.
4. $c = 87.6355$, $b = 26.9148$.
5. $A = 47° 51' 52.1''$, $B = 42° 8' 7.9''$, $b = 184.706$.
6. $A = 55° 43' 57.5''$, $B = 34° 16' 2.5''$, $c = 411.487$.
7. $c = 230.535$, $b = 167.593$.
8. $b = 4023.93$, $a = 36987.4$.
9. $A = 36° 46' 56.7''$, $B = 53° 13' 3.3''$, $c = 41.2042$.
10. $c = 53.5747$, $a = 24.3918$.
11. $A = 65° 30' 18.9''$, $B = 24° 29' 41.1''$, $a = 3.15341$.
12. $A = 58° 35' 43.8''$, $B = 31° 24' 16.2''$, $a = .0409003$.
13. $c = 456.429$, $a = 410.254$.
14. $b = .136736$, $a = .713415$.
15. $c = 1876.87$, $b = 76.1761$.
16. $a = 30.5136$, $b = 18.5891$.
17. $A = 30° 17' 24.4''$, $B = 59° 42' 35.6''$, $c = 20.0382$.
18. $c = 213.637$, $a = 176.603$.
19. $A = 58° 39' 1.6''$, $B = 31° 20' 58.4''$, $b = 1.04098$.
20. $c = 7005.71$, $b = 5571.62$.

21. 14.1069 in.

22. 78.1286 ft.

23. 99.4563 miles.

24. 11.3715 in.

25. 399.386 ft.

26. .773 miles.

27. 26.1313 in. and 24.1422 in.

28. 453.702 ft.

29. 389.421 ft.

30. 5.77349 in.

Art. 142; page 113.

1. $A = 0^\circ\, 0'\, 17.6''$, $B = 89^\circ\, 59'\, 42.4''$.

2. $A = 0^\circ\, 36'\, 38.4''$, $B = 89^\circ\, 23'\, 21.6''$.

3. $A = 89^\circ\, 59'\, 35.8''$, $B = 0^\circ\, 0'\, 24.2''$.

4. $A = 0^\circ\, 0'\, 2.4''$, $B = 89^\circ\, 59'\, 57.6''$.

5. $A = 89^\circ\, 43'\, 13.6''$, $B = 0^\circ\, 16'\, 46.4''$.

Art. 143; page 115.

2. 62432.5.

3. .133216.

4. 5820.45.

5. 6.08685.

6. 31.4775.

7. .000108606.

8. 686128.

9. .206587.

10. 2.29225.

Art. 164; pages 137–139.

1. $b = 14.5843$, $c = 7.09059$.

2. $A = 19^\circ\, 11'\, 17.2''$, $B = 26^\circ\, 37'\, 39''$, $C = 134^\circ\, 11'\, 3.8''$.

3. $A = 33^\circ\, 3'\, 19.7''$, $C = 100^\circ\, 56'\, 40.3''$, $b = 19.782$.

4. $B = 128^\circ\, 49'\, 1.9''$, $C = 19^\circ\, 17'\, 58.1''$, $b = 2.52247$.

5. $B = 37^\circ\, 0'\, 35''$, $C = 79^\circ\, 19'\, 25''$, $b = .11418$;

or, $B = 15^\circ\, 39'\, 25''$, $C = 100^\circ\, 40'\, 35''$, $b = .0511913$.

6. $A = 39^\circ\, 15'\, 56.2''$, $C = 78^\circ\, 46'\, 3.8''$, $b = .320999$.

7. $b = .0345573$, $c = .0189479$.

8. $A = 58^\circ\, 25'\, 47.2''$, $B = 74^\circ\, 23'\, 34.8''$, $C = 47^\circ\, 10'\, 38''$.

9. $b = 8.23709$, $c = 5.46397$.

10. $B = 46^\circ\, 4'\, 24.6''$, $C = 101^\circ\, 6'\, 35.4''$, $c = 91.037$;

or, $B = 133^\circ\, 55'\, 35.4''$, $C = 13^\circ\, 15'\, 24.6''$, $c = 21.275$.

11. $A = 83^\circ\, 14'\, 37.2''$, $B = 45^\circ\, 9'\, 22.8''$, $c = 3.83539$.

12. $A = 55^\circ\, 55'\, 49''$, $B = 79^\circ\, 44'\, 12.2''$, $C = 44^\circ\, 19'\, 58.8''$.

13. $b = 1.00425$, $c = 2.32217$.

14. $A = 68^\circ\, 48'\, 25.2''$, $B = 67^\circ\, 8'\, 34.8''$, $a = 6.92084$;

or, $A = 23^\circ\, 5'\, 34.8''$, $B = 112^\circ\, 51'\, 25.2''$, $a = 2.91143$.

15. $A = 120^\circ\, 17'\, 13.4''$, $C = 24^\circ\, 31'\, 30.6''$, $a = 1.06344$.

16. $a = 447.366$, $b = 425.664$.

17. $A = 52^\circ\, 10'\, 27.3''$, $B = 72^\circ\, 53'\, 32.7''$, $c = .310864$.

18. $a = 342.602$, $c = 303.276$.

19. $A = 49^\circ\, 23'\, 59.6''$, $B = 58^\circ\, 38'\, 11.4''$, $C = 71^\circ\, 57'\, 49.2''$.

20. $A = 90^\circ$, $C = 59^\circ\, 57'$, $c = 8.6559$.

21. $A = 2^\circ\, 39'\, 31.1''$, $B = 171^\circ\, 4'\, 18.9''$, $c = .0838837$.

22. $a = 110.677$, $b = 60.6656$.

23. $B = 37^\circ\, 23'\, 24.6''$, $C = 110^\circ\, 10'\, 49.4''$, $a = 65.352$.

24. $C = 90^\circ$, $B = 49^\circ\, 59'\, 39''$, $b = 127.316$.

25. Impossible.

26. $A = 60^\circ\, 51'\, 3.4''$, $B = 46^\circ\, 7'\, 1.4''$, $C = 73^\circ\, 1'\, 55''$.

27. $A = 67^\circ\, 36'\, 57.6''$, $C = 68^\circ\, 3'\, 9.4''$, $a = 283.908$;

or, $A = 23^\circ\, 43'\, 16.4''$, $C = 111^\circ\, 56'\, 50.6''$, $a = 123.519$.

28. 5112.25.

29. 2624.07.

30. 3506.815.

31. 135.3545.

32. 21595.4.

33. 12.7266.

34. 46.17725.

35. .00181672.

36. .117655.

37. 482.089 ft.

38. 153.629 ft.

39. 29799.8 sq. ft.

40. 247.998 ft.

41. 1569.952 sq. rds.

42. 247.741 ft.

43. *PA*, 8347.42 ft.; *PB*, 7566.65 ft.; *PC*, 5091.92 ft.

Art. 167; page 143.

2. $x = 2.11491,\ -1.86081,$ or $-.254101$.

3. $x = 2.66907,\ -2.1451,$ or $-.523977$.

4. $x = .47762,\ -6.1364,$ or $-.34129$.

5. $x = 3.49086,\ -.83424,$ or $.343382$.

Art. 229; page 189.

5. $A=149^\circ 39' 35.3''$, $B=66^\circ 41' 51.3''$, $b=38^\circ 27' 7.8''$.
6. $a=40^\circ 42' 45.3''$, $b=134^\circ 30' 30.6''$, $c=122^\circ 5' 51.4''$.
7. $A=149^\circ 40' 4.3''$, $B=66^\circ 42' 30.5''$, $c=137^\circ 22' 14.6''$.
8. $a=25^\circ 16' 54.9''$, $c=69^\circ 50' 39''$, $A=27^\circ 3' 38.3''$;

or, $a=154^\circ 43' 5.1''$, $c=110^\circ 9' 21''$, $A=152^\circ 56' 21.7''$.

9. $a=110^\circ 48' 5.3''$, $b=113^\circ 25' 33.6''$, $A=109^\circ 13' 3''$.
10. $A=66^\circ 15' 34.7''$, $b=127^\circ 14' 4.2''$, $c=107^\circ 0' 44.5''$.
11. $A=72^\circ 28' 58.8''$, $B=140^\circ 39' 53.9''$, $c=112^\circ 39' 8.4''$.
12. $b=154^\circ 40' 6.5''$, $B=152^\circ 57' 42''$, $c=70^\circ 15' 13.2''$;

or, $b=25^\circ 19' 53.5''$, $B=27^\circ 2' 18''$, $c=109^\circ 44' 46.8''$.

13. $a=156^\circ 30' 31.4''$, $A=120^\circ 51' 19.7''$, $B=34^\circ 0' 7.1''$.
14. $a=41^\circ 5' 37''$, $b=26^\circ 24' 16.2''$, $c=47^\circ 32' 40.9''$.
15. $b=109^\circ 51' 3.4''$, $B=101^\circ 15' 36.2''$, $c=73^\circ 32' 38.7''$;

or, $b=70^\circ 8' 56.6''$, $B=78^\circ 44' 23.8''$, $c=106^\circ 27' 21.3''$.

16. $a=146^\circ 31' 31.2''$, $b=109^\circ 48' 10.9''$, $c=73^\circ 35' 5.9''$.
17. $A=101^\circ 48' 49.6''$, $b=166^\circ 8' 37.9''$, $c=50^\circ 18' 25.7''$.
18. $a=60^\circ 31' 5''$, $b=147^\circ 31' 53.8''$, $B=143^\circ 50' 6.6''$.
19. $A=112^\circ 2' 35.4''$, $B=109^\circ 12' 6.8''$, $c=81^\circ 53' 36''$.
20. $a=68^\circ 17' 28.5''$, $c=105^\circ 44' 3.8''$, $B=135^\circ 2' 40''$.
21. $a=41^\circ 6' 44.4''$, $b=26^\circ 25' 11.9''$, $B=37^\circ 4' 29.7''$.
22. $a=74^\circ 7' 14.9''$, $A=75^\circ 5' 55.8''$, $B=70^\circ 1' 35.5''$.

Art. 247; pages 203–206.

1. $a=69^\circ 34' 54.1''$, $b=120^\circ 30' 31.5''$, $C=50^\circ 30' 2.6''$.
2. $A=142^\circ 32' 37.8''$, $B=27^\circ 52' 36''$, $C=32^\circ 26' 52.8''$.
3. $a=144^\circ 34' 54.4''$, $b=149^\circ 11' 40.8''$, $c=41^\circ 10' 46.4''$.
4. $a=100^\circ 50' 0.4''$, $A=97^\circ 26' 56.4''$, $C=65^\circ 32' 58.3''$.
5. $A=14^\circ 23' 5.7''$, $C=163^\circ 35' 53.1''$, $b=38^\circ 43' 48.2''$.
6. $a=65^\circ 16' 35.1''$, $c=162^\circ 20' 37''$, $C=165^\circ 41' 37.4''$;

or, $a=114^\circ 43' 24.9''$, $c=100^\circ 7' 37.6''$, $C=126^\circ 40' 40''$.

7. Impossible.
8. $B=42^\circ 37' 26.2''$, $C=160^\circ 1' 41.4''$, $c=153^\circ 39' 8.4''$;
or, $B=137^\circ 22' 33.8''$, $C=50^\circ 19' 1.8''$, $c=90^\circ 5' 12.2''$.
9. $A=35^\circ 30' 54''$, $B=24^\circ 42' 28''$, $C=138^\circ 24' 38.2''$.
10. $A=50^\circ 16' 21.5''$, $B=134^\circ 51' 45.1''$, $c=69^\circ 23' 5.4''$.
11. $b=114^\circ 27' 1.7''$, $a=82^\circ 33' 3.6''$, $A=79^\circ 9' 59.6''$.
12. Impossible.
13. $a=69^\circ 29' 30.7''$, $c=120^\circ 10' 2.7''$, $B=50^\circ 13' 28''$.
14. $C=90^\circ$, $B=113^\circ 38' 46.3''$, $b=114^\circ 53' 15.7''$.
15. $a=89^\circ 25' 36''$, $b=85^\circ 4' 2.8''$, $c=72^\circ 1' 5''$.
16. Impossible.
17. $c=90^\circ$, $b=65^\circ 32' 31.2''$, $B=62^\circ 51' 42.6''$.
18. $b=33^\circ 58' 18.9''$, $c=64^\circ 6' 3.7''$, $A=37^\circ 39' 44''$.
19. $a=100^\circ 0' 8.4''$, $b=49^\circ 59' 56.4''$, $c=60^\circ 0' 11.2''$.
20. $B=42^\circ 37' 21.8''$, $a=153^\circ 38' 48.6''$, $A=160^\circ 1' 26.6''$;
or, $B=137^\circ 22' 38.2''$, $a=90^\circ 5' 43''$, $A=50^\circ 19' 2.6''$.
21. $B=135^\circ 5' 15.2''$, $C=50^\circ 29' 54.8''$, $a=69^\circ 34' 45''$.
22. Impossible.
23. $a=42^\circ 37' 5.2''$, $b=129^\circ 41' 19.4''$, $B=89^\circ 54' 26.2''$;
or, $a=137^\circ 22' 54.8''$, $b=19^\circ 58' 23.4''$, $B=26^\circ 21' 6.4''$.
24. $A=46^\circ 31' 9.6''$, $B=41^\circ 40' 14.8''$, $C=125^\circ 31' 51.8''$.
25. Distance, 3275.17 miles; bearing of Boston from Greenwich, N. 71° 38′ 5.2″ W.; of Greenwich from Boston, N. 53° 7′ 14″ E.
26. Distance, 11,012.8 miles; bearing of Calcutta from Valparaiso, S. 64° 20′ 58.4″ E.; of Valparaiso from Calcutta, S. 54° 55′ W.
27. Latitude 49° 58′ 8.2″ N.
28. 6 h., 0 m., 43 s., A.M.; lon. 44° 49′ 18″ W.
29. 14° 57′ 22.4″.
30. 8 h., 2 m., 44 s., A.M.
31. 5 h., 3 m., 26 s., A.M.